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A Result on Fixed Point in Complete G-Metric Space

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ABSTRACT

In this paper, we define a generalized weakly contraction mapping in G-metric space and present a fixed point theorem for such mapping in complete G-metric space which generalizes a result in the recent literature.

KEYWORDS: G-metric space, fixed point, self map, generalized weakly contraction mapping

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INTRODUCTION AND PRILIMINARIES

In 2006, Mustafa and Sims² introduced a notion of generalized metric space called G-metric space. Later, several authors established many fixed point results for self mappings in G-metric spaces under certain contractions.

Definition 1.1. [2] Let X be a non empty set and let $G: X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following properties:

$$G(x, y, z) = 0 \text{ if } x = y = z.$$

$$G(x, x, y) > 0 \text{ for all } x, y \in X, \text{ with } x \neq y.$$

$$G(x, x, y) \leq G(x, y, z) \text{ for all } x, y, z \in X, \text{ with } y \neq z.$$

$$G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots, \text{ (symmetry in all three variables)}$$

$$G(x, y, z) \leq G(x, a, a) + G(a, y, z), \text{ for all } x, y, z, a \in X \text{ (rectangular inequality)}$$

Then the function G is called a **G-metric on X** and the pair (X, G) is called a **G-metric space**.

Example.[2] Let (X, d) be a usual metric space. Then (X, G) is G-metric space, where

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z), \text{ for all } x, y, z \in X.$$

Definition 1.2.[3] Let (X, G) and (X', G') be G-metric spaces and $f: (X, G) \rightarrow (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if for given $\varepsilon > 0$ there exist $\delta > 0$ such that for $x, y \in X, G(a, x, y) < \delta$ implies that $G(fa, fx, fy) < \varepsilon$.

A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 1.3. [3] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X . We say that $\{x_n\}$ is G-convergent to x if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$. That is, for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call x as the limit of the sequence $\{x_n\}$ and we write $x_n \rightarrow x$ as $n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} x_n = x$.

Proposition 1.4. [3] Let (X, G) and (X', G') be G metric spaces. A function $f: X \rightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, $\{fx_n\}$ is G-convergent to $f(x)$ whenever $\{x_n\}$ is G-convergent to x .

Proposition 1.5.[4] Let (X, G) be a G-metric space. Then the following statements are equivalent:

$\{x_n\}$ is G-convergent to x .

$$G(x_n, x_n, x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$G(x_n, x, x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$G(x_n, x_m, x) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Definition 1.6.[3] Let (X, G) be a G -metric space. A sequence $\{x_n\}$ is called G -Cauchy sequence if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Definition 1.7.[3] A G -metric space (X, G) is said to be G -complete if every G -cauchy sequence in (X, G) is G -convergent in (X, G) .

Proposition 1.8. [4] Every G -metric space (X, G) defines a metric space (X, d_G) , where d_G defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Proposition 1.9. [2] Let (X, G) be a G -metric space. Then for any $x, y, z, a \in X$, the following hold.

$$\text{If } G(x, y, z) = 0 \text{ then } x = y = z.$$

$$G(x, y, z) \leq G(x, x, y) + G(x, x, z).$$

$$G(x, y, y) \leq 2 G(y, x, x).$$

$$G(x, y, z) \leq G(x, a, z) + G(a, y, z).$$

$$G(x, y, z) \leq \frac{2}{3} \{G(x, a, a) + G(y, a, a) + G(z, a, a)\}.$$

Definition 1.10. [5] A mapping $\psi: [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if ψ is continuous and non-decreasing and $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.11. [1] A self mapping $T: X \rightarrow X$, where (X, d) is a metric space, is said to be a generalized weakly contractive mapping if

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(\max\{d(x, y), d(y, Ty)\}) \text{ for all } x, y \text{ where } M(x, y) =$$

$$\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}, \psi \text{ is an altering distance function and}$$

$$\phi: [0, \infty) \rightarrow [0, \infty) \text{ is continuous and non-decreasing function such that } \phi(t) = 0 \text{ if and only if } t = 0.$$

B.S.Choudhury et al. proved the following theorem in ¹.

Theorem 1.12. Let (X, d) be a complete metric space and T be a generalized weakly contractive self mapping of X . Then T has a unique fixed point.

Motivated by the above result, we prove the same result in G -metric spaces.

Definition 1.13. A mapping $T: X \rightarrow X$, where (X, G) is a G -metric space, is said to be a generalized weakly contractive mapping if

$$\varphi(G(Tx, Ty, Tz)) \leq \varphi(M(x, y, z)) - \phi(\max\{G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz)\})$$

for all $x, y, z \in X$

$$(1.14)$$

where $M(x, y, z) = \max \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz),$
 $\frac{1}{3}\{G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)\},$

φ is an altering distance function and $\emptyset: [0, \infty) \rightarrow [0, \infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

MAIN RESULT

Theorem 2.1. A generalized weakly contractive self mapping of a complete G -metric space (X, G) has a unique fixed point.

Proof : Let $x_0 \in X$ be arbitrary . Then we can define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \geq 1$.

If $x_n = x_{n-1}$ for some n , then T has a fixed point.

So assume that $x_n \neq x_{n-1}$ for all $n \geq 1$

Now, from (1.14) we have for all $n \geq 1$

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \varphi(M(x_n, x_{n+1}, x_{n+1})) - \emptyset \left(\max \left\{ G(x_n, x_{n+1}, x_{n+1}), G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_n, Tx_{n+1}, Tx_{n+1}), \right\} \right) \quad (2.2)$$

where,

$$M(x_n, x_{n+1}, x_{n+1}) = \max \left\{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \frac{1}{3}(G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)) \right\} = \max \left\{ \begin{array}{l} G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_{n-1}, Tx_n, Tx_n), \\ G(Tx_n, Tx_{n+1}, Tx_{n+1}), \\ \frac{1}{3}(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_n, Tx_n)) \end{array} \right\} = \max \left\{ G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}), \frac{1}{3}(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_n, Tx_n)) \right\}$$

$$\text{But } \frac{1}{3}(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_n, Tx_n)) \leq \max \{ G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}) \}$$

So $M(x, y, z) = \max \{ G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}) \}$

Now from (1.14), we have

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \varphi(\max\{G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1})\}) - \varphi(\max\{G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1})\}). \quad (2.3)$$

Suppose $G(Tx_{n-1}, Tx_n, Tx_n) \leq G(Tx_n, Tx_{n+1}, Tx_{n+1})$ for some integer n.

Then from (2.3), we have

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) - \varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1}))$$

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq 0 \text{ implies}$$

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) = 0 \text{ which gives}$$

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0, \text{ that is, } G(x_{n+1}, x_{n+2}, x_{n+2}) = 0$$

or $x_{n+2} = x_{n+1}$, a contradiction.

Hence $G(Tx_n, Tx_{n+1}, Tx_{n+1}) < G(Tx_{n-1}, Tx_n, Tx_n) \forall n \geq 1$

Thus $\{G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$ is monotonic decreasing sequence and hence is converging to some real number $r \geq 0$.

$$\text{i.e., } \lim_{n \rightarrow \infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = r$$

Suppose $r > 0$.

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq \varphi(G(Tx_{n-1}, Tx_n, Tx_n)) - \varphi(G(Tx_{n-1}, Tx_n, Tx_n)).$$

Letting $n \rightarrow \infty$, we get $\varphi(r) \leq \varphi(r) - \varphi(r)$ implies

$$\varphi(r) \leq 0, \text{ a contradiction.}$$

Hence $r = 0$.

$$\text{or } \lim_{n \rightarrow \infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0.$$

$$\text{i.e., } \lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0 \quad (2.4)$$

We now show that the sequence $\{x_n\}$ is Cauchy.

Suppose $\{x_n\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ with

$n(k) > m(k) > k$ for all positive integers k such that $G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon$ and

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon \quad (2.5)$$

Now, by rectangular inequality, (2.4) and (2.5), we obtain

$$\varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)})$$

$$\text{So } \varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) < \varepsilon + G(x_{m(k)-1}, x_{m(k)}, x_{n(k)})$$

Letting $k \rightarrow \infty$ in the above inequalities, we get

$$\varepsilon \leq \lim_{k \rightarrow \infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) < \varepsilon + 0$$

$$\text{Therefore } \lim_{k \rightarrow \infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \quad (2.6)$$

Again by rectangular inequality, we have

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)})$$

and

$$G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})$$

Letting $k \rightarrow \infty$, we get from (2.4) and (2.6),

$$\lim_{k \rightarrow \infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon$$

Again by rectangular inequality,

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \text{ and } G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) .$$

Letting $k \rightarrow \infty$ in the above inequalities, we get

$$\lim_{k \rightarrow \infty} G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon.$$

Similarly, we can show that

$$\lim_{k \rightarrow \infty} G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) = \varepsilon.$$

Now, put $x = x_{m(k)}$, $y = x_{n(k)}$, $z = x_{n(k)}$ in (1.14)

$$\varphi(G(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)})) \leq \varphi(M\{G(x_{m(k)}, x_{n(k)}, x_{n(k)})\}) - \varphi\left(\max\left\{G(x_{m(k)}, x_{n(k)}, x_{n(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})\right\}\right) \quad (2.7)$$

where

$$\begin{aligned} &M(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &= \max\left\{G(x_{m(k)}, x_{n(k)}, x_{n(k)}), G(x_{m(k)}, Tx_{m(k)}, Tx_{m(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), \right. \\ &G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), \frac{1}{3}\left(G(x_{m(k)}, Tx_{n(k)}, Tx_{n(k)}) + G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})\right) \\ &\left. + G(x_{n(k)}, Tx_{m(k)}, Tx_{m(k)})\right\} \\ &= \max\left\{G(Tx_{m(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}), G(Tx_{m(k)-1}, Tx_{n(k)}, Tx_{n(k)}), \right. \\ &G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)}), \frac{1}{3}\left(G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)})\right) \\ &\left. + G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)}) + G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)})\right\} \end{aligned}$$

Letting $k \rightarrow \infty$, and using (2.4), (2.6), we get

$$\lim_{k \rightarrow \infty} M(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \max\{\varepsilon, 0, 0, \frac{1}{3}(\varepsilon + 0 + \varepsilon)\} = \varepsilon$$

Now, taking limit as $k \rightarrow \infty$ in (2.6), we get

$$\varphi(\varepsilon) \leq \varphi(\varepsilon) - \emptyset(\varepsilon) \text{ implies } \emptyset(\varepsilon) \leq 0, \text{ a contraction.}$$

Thus $\{x_n\}$ is Cauchy in X . Since X is a complete G - metric space, we can find a $t \in X$ such that $\lim_{n \rightarrow \infty} x_n = t$.

We shall now show that t is a fixed point of T .

Put $x = x_n, y = t, z = t$ in (1.14), we get

$$\begin{aligned} & \varphi(G(Tx_n, Tt, Tt)) \\ & \leq \varphi(M\{G(x_n, t, t)\} - \emptyset(\max\{G(x_n, t, t), G(t, Tt, Tt), G(t, Tt, Tt)\})) \\ & = \varphi \max\left\{G(x_n, t, t), G(x_n, Tx_n, Tx_n), G(t, Tt, Tt), G(t, Tt, Tt), \right. \\ & \quad \left. \frac{1}{3}(G(x_n, Tt, Tt) + G(t, Tt, Tt) + G(Tt, Tx_n, Tx_n))\right\} \\ & \quad - \emptyset \max\{G(x_n, t, t), G(t, Tt, Tt)G(t, Tt, Tt)\} \end{aligned}$$

$$\varphi(G(x_{n+1}, Tt, Tt)) \leq$$

$$\varphi\left(\max\left\{0, G(t, Tt, Tt), \frac{2}{3}(G(t, Tt, Tt))\right\}\right) - \emptyset \max\{G(t, t, t), G(t, Tt, Tt)\}$$

Taking limit as $n \rightarrow \infty$,

$$\varphi(G(t, Tt, Tt)) \leq \varphi(G(t, Tt, Tt)) - \emptyset(G(t, Tt, Tt))$$

So $G(t, Tt, Tt) = 0$ which gives $Tt = t$. This shows that t is a fixed point of T .

To prove uniqueness, let t^1 be another fixed point of T .

Put $x = t, y = t^1, z = t^1$ in (1.14), we get

$$\varphi(G(Tt, Tt^1, Tt^1)) \leq \varphi(M(t, t^1, t^1)) - \emptyset(\max(G(t, t^1, t^1), G(t^1, Tt^1, Tt^1), G(t^1, Tt^1, Tt^1)))$$

$$\text{where } M(t, t^1, t^1) = \max\{G(t, t^1, t^1), G(t, Tt, Tt), G(t^1, Tt^1, Tt^1), G(t^1, Tt^1, Tt^1), \frac{1}{3}(G(t, Tt^1, Tt^1) + G(t^1, Tt^1, Tt^1) + G(t^1, Tt, Tt))\}$$

$$= \max\left\{G(t, t^1, t^1), 0, 0, 0, \frac{1}{3}(G(t, t^1, t^1) + G(t^1, t^1, t^1) + G(t^1, t, t))\right\}$$

$$= G(t, t^1, t^1), \text{ since } G(t^1, t, t) \leq 2G(t, t^1, t^1)$$

$$\text{So } \varphi(G(t, t^1, t^1)) \leq \varphi(G(t, t^1, t^1)) - \emptyset(G(t, t^1, t^1))$$

which implies $G(t, t^1, t^1) = 0$ and therefore $t = t^1$.

Corollary 2.8. Suppose (X, G) is a complete G -metric space. If the

mapping $T: X \rightarrow X$ satisfies the following condition $\forall x, y, z \in X$
 $\varphi(G(Tx, Ty, Tz)) \leq$

$\varphi(\max \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}) -$

$\emptyset(\max\{G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz)\})$, where

φ is an altering distance function and $\emptyset: [0, \infty) \rightarrow [0, \infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

Then T has a unique fixed point in X.

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