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Some Characterizations of Strictly Convex 2- Normed Spaces

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ABSTRACT:

The concept 2 normed space was introduced by S. Gahler. After Daminni and A.White introduced the ideas of strictly 2 convex 2 –normed space. They gave several important result on strictly convex 2-nornmed spaces. This paper consist of several characterization of strictly convex 2 normed Spaces.

KEYWORDS:

2 normed space, Strictly convex, 2 dimensional analog.

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INTRODUCTION:

Let L, be a linear space of dimension greater than L and $\|\cdot,\cdot\|$ a real valued function on L XL which satisfies the following four conditions:

1. $\| a,b \| = 0$ if and only if a and b are linearly dependent,
2. $\| a,b \| = \| b, a \|$,
3. $\| a,\beta b \| = \| \beta \| \| a,b \|$, when β is real,
4. $\| a,b+c \| \leq \| a,b \| + \| a,c \|$.

$\|\cdot,\cdot\|$ is called a 2- norm on L and (I, $\|\cdot,\cdot\|$) linear 2- normed space.

The 2- norm is a non-negative function.

With respect to the definition of the 2-norm the notion of Linear 2- normed space is a 2- dimensional analog to the notion of normed linear space. A normed linear space is called strictly convex if $\| x+y \| = \| x \| + \| y \|$ and $\| x \| = \| y \| = 1$ imply that x = y.

For non-zero vectors $a,b \in L$, let V (a, b) denote the subspace of L generated by a and b. whenever the notation V (a, b) is used, it will be understood that a and b are non-zero vectors. A linear 2 – normed space (L, $\|\cdot,\cdot\|$) said to be strictly convex if $\| a+b,c \| + \| \ b,c \|, \| a,c \| = \| b,c \| = 1$ and $c \notin V(a,b)$ imply that a = b.

If c is a fixed non-zero element of L, let V(c) denote the linear subspace of L generated c and let L, be the quotient space L/V(c). For $a \in L$, let a_c represent the equivalence class of a with respect to V(c). Le is a vector space with addition given by $a_c + b_c = (a+b)_c$ and scalar multiplication by $\alpha a_c = (\alpha a)_c$. For arbitrary $a,b \in L$ which satisfy $a_c = b_c$, the conditions.

$\| a,c \| - \| b,c \| \ \| a-b,\ c \| = 0,$ and thus, $\| a,\ c \| - \| b,\ c \|$. Therefore, the real-valued functions $\|\cdot\|_c$ on L_c given by $\| a_c \|_c = \| a,c \|$ is well-normed.

Lemma:

$\|\cdot\|_c$ is norm on L_c.

Proof:

1. $\| a_c \|_c = 0$ if and only if $\| a,\ c \| = 0$ i.e., if and only if $a_c = 0_c$
2. $\| \alpha a_c \|_c = \| (\alpha a)_c \|_c = \| \alpha, a, c \| = | \alpha \| \ a,c \| = | \alpha \| \| a,c \| = | \alpha | \| ac \|$, when α is real.

3. $\| a_c + b_c \|_c = \| (a+b)_c \|_c = \| a+b, c \| \le \| a, c \| + \| b, c \| = \| a_c \|_c + \| b_c \|_c$.

Theorem :

For a linear 2-normed space (L, ‖·,·‖), the following are equivalent :

1. (L, ‖·,·‖) is strictly convex.
2. For every non-zero $c \in L, (L_c, \|\cdot\|)$ is strictly convex.
3. $\| a+b, c \| = a, b \| + \| b, c \|$ and $c \in V(a,b)$ imply that $a = \alpha a,$ for some $\alpha > 0.$
4. $\| a-d, c \| = \alpha \| a-b, c \|, \| b-d, c \| = (a-\alpha) \| a-b, c \| \alpha \in (0,1),$ and $c \in V(a-d, b-d)$ imply that $d = (1-\in) a + \alpha b.$

Proof:

1. Let (L, ‖·,·‖) be strictly convex and let c be a fixed non-zero element of L. If $\| a_c + b_c \|_c = \| a_c \|_c + \| b_c \|$ and $\| a_c \|_c = \| b_c \|_c = I,$ then $\| a+b, c \| = \| a, c \| + \| b, c \|$ and $\| a, c \| = \| b, c \| = 1.$ For the case $c \in V(a,b)$ and α and β then $0_c + c_c = \alpha a_c + \beta b_c$ or $\alpha a = -\beta b_c.$ From $\| a_c \|_c = \| b_c \|_c = I,$ it follows that $\alpha = \beta$ If $\alpha = \beta$ then $c = \alpha(a+b)$ which contradicts $\| a+b, c \| =$ $\|a_c + b_c\|_c = 2$. Therefore, $\alpha = -$ and hence $a_c = b_c$. Thus (L_c, ||·|| is strictly convex.
2. Assume condition 2 holds. For c $\in$V (a, b), let ||a + b, c|| = ||a, c|| + ||b, c||, i.e., $\| a_c + b_c \|_c = \| a_c \|_c + \| b_c \|_c$ By the strict convexity of $(L_c \|\cdot\| e),$ it follows that $b_c = \alpha a_c,$ some $\alpha > 0.$ Finally, $c \in V(a,b)$ implies that $b = \alpha a$ and condition 3 is satisfied.
3. Assume condition 3. From $\| a-d, c \| = \alpha \| a-b, c \| = 1-\alpha \| a-b, c \|,$ $c \in V(a-d, b-d)$ and $\alpha \in (0,1),$ it follows that $(1-\alpha) \| a-b, c \| = \| b-d, c \|$ = $\beta \| a-d, c \| = \alpha\beta \| a-b, c \|$ implies that $d = (1-\alpha)\alpha a + b.$ Therefore, condition 4 is established.
4. Finally, assume condition 4 and let $\| a+b, c \| = \| a, c \| + \| b, c \|, \| a, c \| = \| b, c \| = 1$, where $c \in V(a,b).$ By condition 4, 0 = a – b, i.e. a = b. Therefore (L, ‖·,·‖) is strictly convex and the theorem is proven.

If L is a linear space of dimension greater then 1, let B'_1 be the set of all formal expression $\sum a_i \times b_i,$ where a_i, b_i (i = 1, …., m) vectors in L. Let – be the equivalent relation on B'_1 defined by

$$\sum_{i=1}^{m} a_i \times b_i = \sum_{i=1}^{m} a_i \times b_i$$

if for arbitrary linear functions f and g on L,

$$\sum_{i=1}^{m}\begin{vmatrix} f(a_i) & g(a_i) \\ f(b_i) & b(b_i) \end{vmatrix} = \sum_{i=1}^{m}\begin{vmatrix} f(a_i^1) & g(a_i^1) \\ f(b_i^1) & g(b_i^1) \end{vmatrix}$$

Let B_1 be the quotient space B_1/–. The elements of B_1 are called bivectors over L and the elements of $B_1^{'}$ belonging to a bivector are called representatives of this bivector. The bivector with the representative $\sum_{i=1}^{m} a_i \times b_i$ will also be denoted by $\partial\left(\sum_{i=1}^{m} a_i \times b_i\right)$. If a bivector has a representative of the form $\sum_{i=1}^{m} a_i \times b_i = a_i \times b_i$, then it is said to be simple. Only in the case where L has dimension less than or equal to 3 does every bivector over L turn out to be simple. The space B_1 is a linear space with $\left(\sum_{i=1}^{m} a_i \times b_i\right) + \left(\sum_{i=1}^{m} a_{i+m} \times b_{i+m}\right) = \left(\sum_{i=1}^{m+n} a_i \times b_i\right)$ and $\beta\partial\left(\sum_{i=1}^{m} a_i \times b_i\right) = \partial\left(\sum_{i=1}^{m} a_i \times \beta b_i\right)$, when β is real. If $\|\cdot\|$ is a norm on B_L, then $\| a, b \| = \| (a \times b) \|$ defines a 2– norm on L. There is an example which shows that for every 2– norm $\|\cdot,\cdot\|$ on L, there need not exist a norm $\|\cdot\|$ on B_1 which satisfies $\| (a \times b) \| - \| a\, b \|$ for all $a, b \in L$ if bivector L, i.e., if L has dimension less than or equal to 3, then for every 2 – norm $\|\cdot,\cdot\|$ on L there is a norm $\|\cdot\|$ on B_1 with $\| (a \times b) \| - \| a\, b \|$ for all $a, b \in L$.

Theorem :

Let L be a linear space of dimension greater than 1, $\|\cdot\|$, be a norm on B_1, and $\|\cdot,\cdot\|$ be a 2– norm on L with $\| (a \times b) \| = \| a, b \|$ for all $a, b \in L$. If $(B_1, \|\cdot\|)$ is strictly convex, then (L, $\|\cdot,\cdot\|$) is strictly convex. If the dimension of L is less than or equal to 3 and (L, $\|\cdot,\cdot\|$) is strictly convex, then $(B_1, \|\cdot\|)$ is strictly convex.

Proof:

1. Suppose $(B_1, \|\cdot\|)$ is strictly convex 1 or $c \in V(a, b)$, assume $\| a + b, c \| = \| a, c \| + \| b, c \|$ and $\| a, c \| = \| b, c \| = 1$. Then, $\| (a \times c) + (b \times c) \| = \| (a \times c) \| + \| (b \times c) \|$ and $\| {*}(a \times c) \| = {*}(b \times c) \| = 1$.

Since $(B_1,\|\cdot\|)$ is strictly convex, it follows that $*(a\times c)\|=*(b\times c)$. This implies that $a-b\in V(c)$. Therefore, a = b, since $c\notin V(a,b)$.

2. Suppose L has dimension less than or equal to 3 and (L, $\|\cdot,\cdot\|$) is strictly convex. Let $\|*1+*2\|=\|*1\|+\|*2$ and $\|*1=\|*2\|=1$. Hence, there exist vectors a, b, $c\in L$ with $*1=*(a\times c)$ and $*2=*(b\times c)$. Thus, $\|a+b,c\|=\|a,c\|+\|b,c\|$ and $\|a,c\|=\|b,c\|=1$. If $c\in V(a,b)$, these equations imply that $c=\alpha(a-b)$ for some real, and hence *1 = *2. If $c\notin V(a,b)$, then strict convexity of (L, $\|\cdot,\cdot\|$) implies that a = b, i.e. *1 = *2.

If L is a 2–dimensional linear space, then B_1 is a 1–dimensional normal linear space and every 1–dimenstional normed space is trivally strictly convex. Therefore, every 2–dimensional linear 2–normed space is strictly convex. Also, it follows that there are linear 2–normed space which are not strictly convex.

Theorem :

Let (L, $\|\cdot,\cdot\|$) be a linear 2–normed space and $(L',\|\cdot\|)$ be a linear normed space.

Proof:

1. If (L, $\|\cdot,\cdot\|$) is strictly convex, c is a fixed non-zero-element of L, and f is a function from L into L which satisfies $\|f(a)-f(b),c\|=\|a-b\|$ for every $a,b\in L,$ then the function g_c from L' into L_c, defined by $g_c(a)=[-f(a)]_c$, is linear.

   If $(L,\|\cdot\|)$ is strictly convex, c is a fixed non-zero element of L, and f a function from L into L' satisfying $\|f(a)-f(b)\|=\|a-b,c\|,$ for every $a,\in bL,$ then the function g from L into L', defined by $g(a)=f(a)-f(0),$ is linear.

   Let (L, $\|\cdot,\cdot\|$) be strictly convex and f, c, g_c be as given in statement 1. By a known theorem, $(L_c,\|\cdot\|_c)$ is strictly convex. For every $a,b\in L'$,

$$
\begin{aligned}
\|g_c(a)-g_c(b)\|_c &=\|[f(a)-f(b)]_c\|_c\\
&=\|f(a)-f(b),c\|\\
&=\|a-b\|.
\end{aligned}
$$

   If follows that g_c is linear.

2. Let $(L', \|\cdot\|)$ be strictly convex and f, c, g be as given in statement 2. Then $g(0)=0$ and for every a, be L.

$\| g_c(a) - g(b) \| = \| f(a) - f(b) = \| a - b, c \|$

Thus, for $a, b \in L$ which satisfy $a_c - b_c$, it follows that $\| g(a) - g(b) \| = 0$ and the function g_c from L_c into L', given by $g_c(a_c) - g(a)$ is well-defined F or any

$$a, b \in, L = \| g_c(a_c) - g_c(b_c) \| = \| g(a) \| - g(b) \|$$

$$= \| a - b, c \|$$

$$= \| a_c - b_c \|_c$$

Since $g_c(0_c) = g(0) = 0,$ it implies that g, is linear. For any $a, b \in L$ and any real number $g(a) = g_c[(\alpha a)_c] = \alpha g_c(a_c)$, and $g(a+b) = g_c(a_c + b_c) = g_c(a_c) + g_c(b_c) = g(a) + g(b)$

Therefore, g is linear.

COROLLARY:

Let (L, ‖·,·‖) be a strictly convex linear 2–normed space, c be a fixed non-zero element of L, and f be a function from L into L which satisfies $f(0)=0$ and $\| f(a) - f(b), c \| = \| a - b, c \|$ for every $a, b \in L$ Then the function g_c from L into L_c, defined by $g_c(a) = [f(a)_c]$, is linear.

Proof:

Since (L, ‖·,·‖) is strictly convex it implies that $(L_c, \|\cdot\|)$ is strictly convex. For any $a, b \in L$,

$$\| g_c(a) - g_c(b) \|_c = \| [f(a) - f(b)]_c \|_c$$

$$= \| f(a) - f(b), c \|$$

$$= \| a - b, c \|$$

From part 2 of the proceeding theorem, it follows the g_c in linear

Definition:

If M and N are linear subspaces of L, a bilinear form F on M × N is said to be bounded if there is a number K > 0 for which $|F(a,b)| \le K \| a, b \|$ for every $(a,b) \in M \times N$

The norm of F, ||F|| is defined by

$\| F \| = \inf\{K : |F(a,b)| \le K \| a, b \|\}$ for every $(a,b) \in M \times N$.

Theorem:

The following are equivalent :

A. $(L,\|\cdot\|)$ is strictly convex.

B. If $c \neq 0$, F is a non-zero bounded bilinear form on $L \times V(c), \| x,c \| = \| y,c \|$ =1 and $F(x,c) = F(y,c) = \| F \|$, then either x = y or $\| x,y \| \neq 0$ and $c = \pm \frac{1}{\| x,y \|^{(x-y)}}$

Proof:

A. Assume $(L,\|,\|)$ is strictly convex. Let $c \neq 0$ and F be a non-zero bounded bilinear form on $L \times V(c)$. If

$F(x,c) = F(y,c) = \| F \|$ and $\| x,c \| = \| y,c \| = 1$, then $2 = \frac{1}{\| F \|} F(x+y,c) \leq \| x+y,c \|$ $\leq \| x,c \| + \| y,c \| = 2$. Therefore $\| x+y,c \| = 2$. If $x \neq y$, then $c \in V(x,y)$ since otherwise the strict convexity of L would yield x = y. Hence, there are real numbers α and β for which $c = \alpha x + \beta y$.

Then, $1 = \| x,c \| = \| \alpha x + \beta y \| = | \beta | \, \| x,y \|$. Similarly $| \alpha | \, \| x,y \| = 1$.

Therefore, $\| x,y \| \neq 0$ and $| \alpha | = | \beta | = \frac{1}{\| x,y \|}$ Since $\| x+y,c \| = 2$, it follows that $= \pm \frac{1}{\| x,y \|^{(x-y)}}$

B. Assume condition 2 holds and let $\| a,c \| = \| b,c \| = 1, a \neq b$ and $c \notin V(a,b)$.

Then, $\| a+b,\ c \| \leq \| a,c \| + \| b,c \| = 2$. If $\| a+b,\ c \| = 2$, there is a bounded bilinear form F defined on $L \times V(c)$ such that $\| F \| = 1$ and $F\left(\frac{a+b}{2}, c\right) = \left\| \frac{a+b}{2}, c \right\| = 1$

Note that $F(a,c) \leq | F(a,c) | \leq \| F \| \, a,c \| = 1$. If $F(a,c) = 1$, then since $a \neq \frac{a+b}{2}$, condition 2 with x = a and $y = \frac{a+b}{2}$ implies that $c \in V(a,b)$ which is impossible.

Thus $F(a,c) < 1$. A similar argument shows that $F(a,c) < 1$ also. Therefore,

$$1 = \frac{1}{2} F(a+b,c) = \frac{1}{2} F(a+c) + \frac{1}{2} F(b,c) < 1$$

Hence, $\| a+b,c \| < 2$ and $(L,\|,\|)$ is strictly convex.

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