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### **Total Dominator Chromatic Number on Various Classes of Graphs**

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#### **ABSTRACT**

Let  $G$  be a graph with minimum degree at least one. A total dominator coloring of  $G$  is a proper coloring of  $G$  with the extra property that every vertex in  $G$  properly dominates a color class. The total dominator chromatic number of  $G$  is denoted by  $\chi_{td}(G)$  and is defined by the minimum number of colors needed in a total dominator coloring of  $G$ . In this paper, we obtain total dominator chromatic number on various classes of graphs.

**MATHEMATICS SUBJECT CLASSIFICATION: 05C15, 05C69**

**KEYWORDS :** Total dominator chromatic number, banana graph ,book graph, stacked book graph, dutch wind mill graph, lollipop graph, gear graph, sunflower graph.

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## INTRODUCTION

All graphs considered in this paper are finite, undirected graphs and we follow standard definition of graph theory as found in<sup>1</sup>. Let  $G = (V, E)$  be a graph of order  $n$  with minimum degree atleast one . The open neighborhood  $N(v)$  of a vertex  $v \in V(G)$  consists of the set of all vertices adjacent to  $v$ . The closed neighborhood of  $v$  is  $N[v] = N(v) \cup \{v\}$ . An induced subgraph  $G[S]$ , where  $S \subseteq V$  of a graph  $G$  is a graph formed from a subset  $S$  of the vertices of  $G$  and all of the edges connecting pairs of vertices in  $S$ . A graph in which every pair of vertices is joined by exactly one edge is called complete graph. A complete bi partite graph is a graph whose vertices can be partitioned into two subsets  $V_1$  and  $V_2$  such that no edge has both end points in the same subset, and each vertex of the first set is connected to every vertex of the second set and vice -verse. A star graph  $S_n$  is the complete bipartite graph  $K_{1,n-1}$  (A tree with one internal node and  $n-1$  leaves).

The path and cycle of order  $n$  are denoted by  $P_n$  and  $C_n$  respectively. For any two graphs  $G$  and  $H$ , we define the cartesian product, denoted by  $G \times H$ , to be the graph with vertex set  $V(G) \times V(H)$  and edges between two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  iff either  $u_1 = u_2$  and  $v_1 v_2 \in E(H)$  or  $u_1 u_2 \in E(G)$  and  $v_1 = v_2$ .

A subset  $S$  of  $V$  is called a total dominating set if every vertex in  $V$  is adjacent to some vertex in  $S$ . The total dominating set is minimal total dominating set if no proper subset of  $S$  is a total dominating set of  $G$ . The total domination number  $\gamma_t$  is the minimum cardinality taken over all minimal total dominating set of  $G$ . A  $\gamma_t$ -set is any minimal total dominating set with cardinality  $\gamma_t$ .

A proper coloring of  $G$  is an assignment of colors to the vertices of  $G$  such that adjacent vertices have different colors. The minimum number of colors for which there exists a proper coloring of  $G$  is called chromatic number of  $G$  and is denoted by  $\chi(G)$ . A total dominator coloring (td- coloring) of  $G$  is a proper coloring of  $G$  with the extra property that every vertex in  $G$  properly dominates a color class. The total dominator chromatic number is denoted by  $\chi_{td}(G)$  and is defined by the minimum number of colors needed in a total dominator coloring of  $G$ . This concept was introduced by A.Vijiyalekshmi in<sup>2</sup>. This notion is also referred as a smarandachely  $k$  - dominator coloring of  $G$  ( $k \geq 1$ ) and was introduced by A.Vijiyalekshmi in<sup>3</sup>. For an integer  $k \geq 1$ , a smarandachely  $k$ -dominator coloring of  $G$  is a proper coloring of  $G$  such that every vertex in  $G$  properly dominates a  $k$  color class. The smallest number of colors for which there exist a smarandachely  $k$ -dominator coloring of  $G$  is called the smarandachely  $k$ -dominator chromatic number of  $G$ , and is denoted by  $\chi_{td}^s(G)$ .

In a proper coloring  $C$  of  $G$ , a color class of  $C$  is a set consisting of all those vertices assigned the same color. Let  $C^*$  be a minimal td-coloring of  $G$ . We say that a color class  $c_i \in C^*$  is called a non-dominated

color class (n-d color class) if it is not dominated by any vertex of G. These color classes are also called repeated color classes. A banana graph  $B_{m,n}$  is a graph obtained by connecting one leaf of each m copies of an n-star graph with a single root vertex that is distinct from all the stars. The book graph  $B_m$  is defined as the graph Cartesian product  $P_2 \times K_{1,m-1}$ . The stacked book graph  $SB_{m,n}$  is the generalization of the book graph to stacked pages. The dutch windmill graph  $D_m^n$  is the graph obtained by taking n copies of the cycle graph  $C_n$  with a vertex in common. The lollipop graph  $L_{m,n}$  is a graph consisting of a complete graph on m vertices and a path graph on n vertices connected with a bridge. The gear graph  $G_n$  is a wheel graph with a one single vertex added between each pair of adjacent vertices of the outer cycle. A sunflower graph  $Sf_n$ , where  $n \geq 4$  is a graph obtained from n- cycle  $C_n$  by including a triangle on each outer edge so that one vertex of each outer triangle has degree 2.

The total dominator chromatic number of paths, cycles and ladder graphs were found in<sup>4</sup>.

We have the following observations from<sup>4</sup>.

**Theorem A<sup>4</sup>**. Let G be  $p_n$  or  $C_n$ . Then

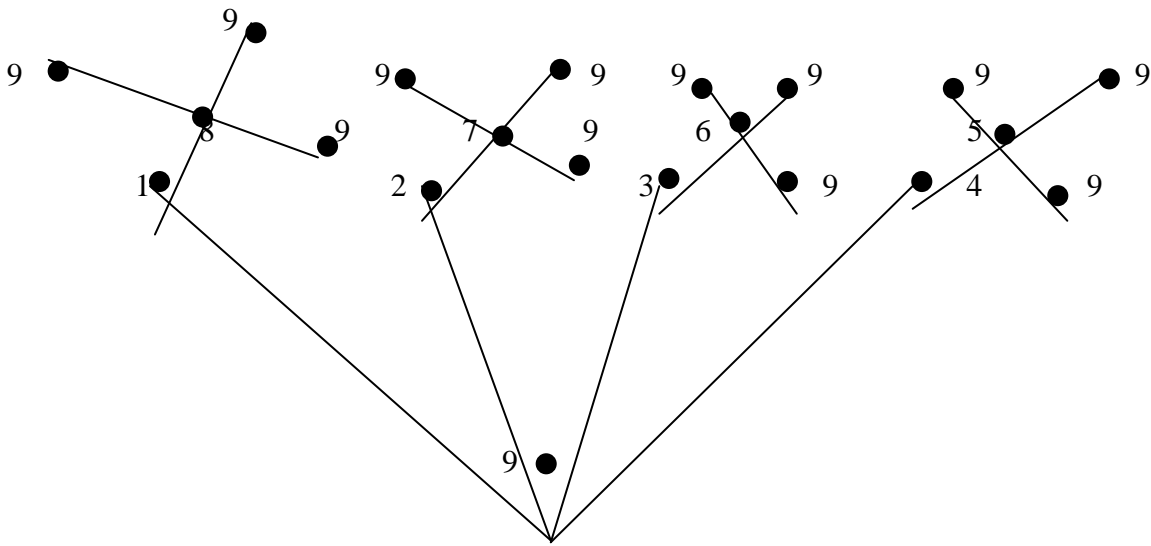
$$\chi_{td}(p_n) = \chi_{td}(C_n) = \begin{cases} 2 \left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{if } n \equiv 0 \pmod{4} \\ 2 \left\lfloor \frac{n}{4} \right\rfloor + 3 & \text{if } n \equiv 1 \pmod{4} \\ 2 \left\lfloor \frac{n+2}{4} \right\rfloor + 2 & \text{otherwise} \end{cases}$$

In this paper, we obtain the least value for total dominator chromatic number on various classes of graphs.

**Theorem 1** For the banana graph  $B_{m,n}$   $\chi_{td}(B_{m,n}) = 2m+1$

**Proof:** Let  $B_{m,n}$  be the banana graph. The vertex set of the graph  $V(B_{m,n}) =$

$\{u\} \cup \{v_{ij} / 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ . That is,  $B_{m,n}$  consist of one vertex has degree m and m vertices of degree 2 and m vertices of degree n-1 and  $m(n-2)$  vertices of degree 1. We assign 2m distinct colors to degree 2 and n-1 respectively and the color say 2m+1 to the vertices of degree 1 and  $\square$  Thus  $\chi_{td}(B_{m,n}) = 2m+1$ .

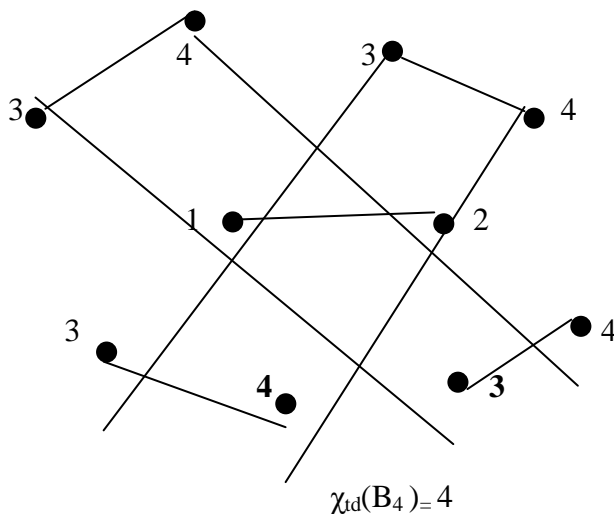


**Fig 1 Banana Graph**

$$\chi_{td}(B_{4,5}) = 9$$

**Theorem 2** For the book graph  $B_m$ ,  $\chi_{td}(B_m) = 4$

**Proof :** Let  $P_2 \times K_{1,m}$  be the book graph with vertex set  $\{v_1, v_2, v_3, \dots, v_{2n}, v_{2n+1}, v_{2n+2}\}$ , where  $(v_1, v_2)$  and  $(v_i, v_j)$   $i=3,5,7,\dots,2n+1$  and  $j=4,6,8,\dots,2n+2$  form the pages of  $B_m$ . We assign colors 1 and 2 to  $v_1$  and  $v_2$  respectively, assign the colors 3 and 4 to the set of vertices  $\{v_3, v_5, v_7, \dots, v_{2n+1}\}$  and the set of vertices  $\{v_4, v_6, v_8, \dots, v_{2n+2}\}$  respectively. Thus  $\chi_{td}(B_m) = 4$ .  $\square$

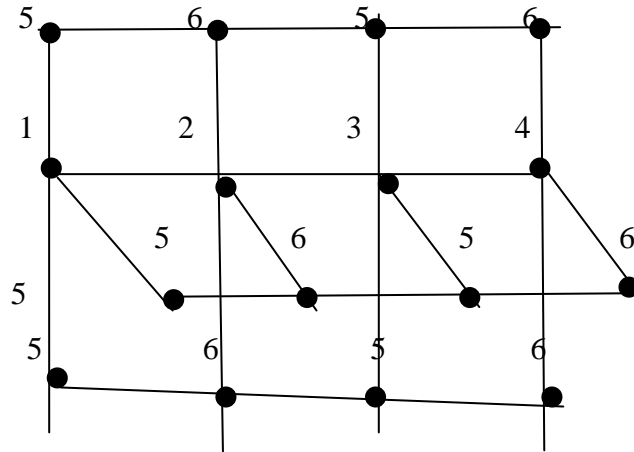


$$\chi_{td}(B_4) = 4$$

**Fig 2 Book Graph**

**Theorem 3** For any stacked book graph  $SB_{m,n}$ ,  $\chi_{td}(SB_{m,n})=n+2$

**Proof:** Let  $SB_{m,n}=P_n \times K_{1,m}$  be the stacked book graph and let  $V(SB_{m,n})= \{v_{ij} / 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$  such that  $B_i$  isomorphic to the vertex induced subgraph  $v_{1i}, v_{2i}, v_{3i}, \dots, v_{ni}$ . We assign  $n$  distinct colors  $1, 2, 3, \dots, n$  to  $v_{11}, v_{21}, v_{31}, \dots, v_{n1}$  and colors  $n+1$  and  $n+2$  to the set of vertices  $v_{ij}, 1 \leq j \leq m$  and  $i=1, 3, 5, \dots, n$  if  $n$  is odd and the set of vertices  $v_{ij}, 1 \leq j \leq m$  and  $i=2, 4, 6, \dots, n$  if  $n$  is even respectively. Thus  $\chi_{td}(SB_{m,n})=n+2$ .



$$\chi_{td}(SB_{3,4})=6$$

**Fig 3** Stacked Book Graph

**Theorem 4** For the dutch wind mill graph  $D_m^n$ ,

$$\chi_{td}(D_m^n) = \begin{cases} n \left( 2 \left\lfloor \frac{m-3}{4} \right\rfloor + 3 \right) - 2n + 4 & \text{if } m \equiv 0 \pmod{4} \\ n \left( 2 \left\lfloor \frac{m-3}{4} \right\rfloor + 2 \right) - 2n + 4 & \text{if } m \equiv 3 \pmod{4} \\ n \left( 2 \left\lfloor \frac{m-1}{4} \right\rfloor + 2 \right) - 2n + 4 & \text{otherwise} \end{cases}$$

**Proof:** Consider  $D_m^n$  formed by  $n$  copies of the cycle  $C_m$  with  $V(D_m^n) = \{v_{ij} / \substack{i=1,2,3,\dots,n \\ j=1,2,3,\dots,m}\}$ . For each  $i=1, 2, 3, \dots, n$   $\{v_{i1}, v_{i2}, v_{i3}, \dots, v_{im}\}$  be the vertices of  $i$ -th copy of cycle  $C_m$  and  $v_{11}=v_{21}=v_{31}=\dots=v_{n1}$  is a common vertex. We assign color 1 and 2 to a common vertex  $v_{11}$  and the set of vertices  $\{v_{i2}, v_{i3}, v_{i4}, v_{i5}, \dots, v_{i,m-1}\}, i=1, 2, 3, \dots, n$  and we assign  $n \chi_{td}(C_{m-3})$  distinct colors to remaining vertices  $\{v_{i3}, v_{i4}, v_{i5}, \dots, v_{i,m-1}\}, i=1, 2, 3, \dots, n$ . Totally we get  $n \chi_{td}(C_{m-3}) + 2 - 2(n-1)$ . We using repeated colour, so  $\chi_{td}(D_m^n) = n \chi_{td}(C_{m-3}) + 2 - 2(n-1)$ .

$$\text{Thus } \chi_{td}(D_m^n) = \begin{cases} n \left( 2 \left\lfloor \frac{m-3}{4} \right\rfloor + 3 \right) - 2n + 4 & \text{if } m \equiv 0 \pmod{4} \\ n \left( 2 \left\lfloor \frac{m-3}{4} \right\rfloor + 2 \right) - 2n + 4 & \text{if } m \equiv 3 \pmod{4} \\ n \left( 2 \left\lfloor \frac{m-1}{4} \right\rfloor + 2 \right) - 2n + 4 & \text{otherwise} \end{cases} \quad \square$$

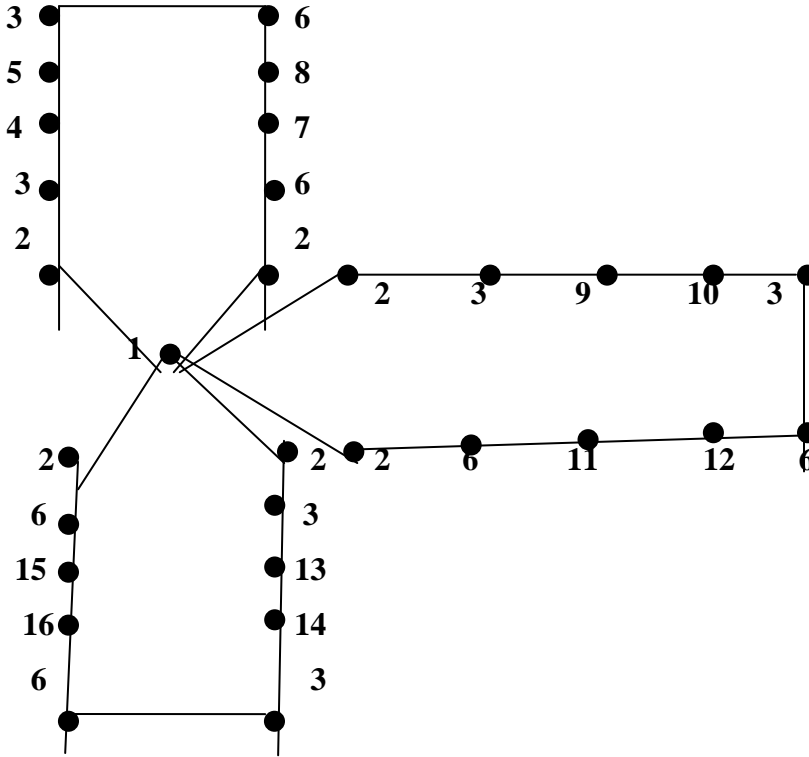


Fig 4 Dutch Wind mill Graph

$$\chi_{td}(D_{11}^3) = 16$$

**Theorem 5** For lollipop graph  $L_{m,n}$ ,

$$\chi_{td}(L_{m,n}) = \begin{cases} 2 \left\lfloor \frac{n-2}{4} \right\rfloor + 2 & \text{if } n \equiv 2 \pmod{4} \\ 2 \left\lfloor \frac{n-2}{4} \right\rfloor + 3 & \text{if } n \equiv 3 \pmod{4} \\ 2 \left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{otherwise} \end{cases}$$

**Proof:** Let  $L_{m,n}$  be the lollipop graph and let  $V(L_{m,n}) = \{v_1, v_2, v_3, \dots, v_m, v_{m+1}, v_{m+2}, v_{m+3}, \dots, v_{m+n}\}$  be the set of vertex set, where the set of vertices  $\{v_1, v_2, v_3, \dots, v_m\}$  form  $K_m$  and the set of vertices  $\{v_{m+1}, v_{m+2}, v_{m+3}, \dots, v_{m+n}\}$  form  $P_n$  and  $(v_m, v_{m+1})$  is a bridge of  $L_{m,n}$ . We assign colors 1 to the vertex set  $\{v_m, v_{m+2}\}$  and the set of vertices  $\{v_1, v_2, v_3, \dots, v_{m-1}, v_{m+1}\}$  receive  $m$  distinct colors say 2, 3, ...,  $m+1$  respectively. Remaining  $(n-2)$  vertices  $\{v_{m+3}, v_{m+4}, v_{m+5}, \dots, v_{m+n}\}$  have  $\chi_{td}(P_{n-2})$  colors for  $td$ -coloring.

$$\text{Thus } \chi_{td}(L_{m,n}) = \begin{cases} 2 \left\lfloor \frac{n-2}{4} \right\rfloor + 2 & \text{if } n \equiv 2 \pmod{4} \\ 2 \left\lfloor \frac{n-2}{4} \right\rfloor + 3 & \text{if } n \equiv 3 \pmod{4} \\ 2 \left\lfloor \frac{n}{4} \right\rfloor + 2 & \text{otherwise} \end{cases}$$

□

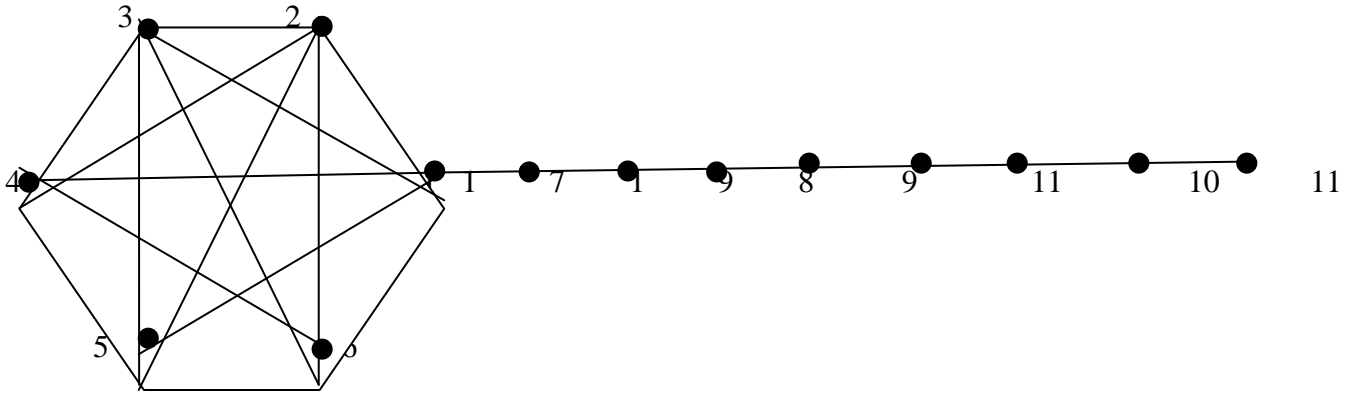


Fig 5 Lollipop Graph

$$\chi_{td}(L_{6,8})=11$$

**Theorem 6** Any gear graph  $G_n$ ,  $\chi_{td}(G_n) = \chi_{td}(C_{2n})$

**Proof:** Let  $G_n$  be the gear graph with vertex set  $\{v_1, v_2, v_3, \dots, v_n, v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n+1}\}$ , where  $v_1$  is the central vertex and  $v_i$  ( $2 \leq i \leq 2n+1$ ) be the vertices on the cycle  $C_{2n}$ . For td-coloring, we need  $\chi_{td}(C_{2n})$  colors for vertex set  $\{v_i / 2 \leq i \leq 2n+1\}$  and the central vertex receive any one of the above color. Thus  $\chi_{td}(G_n) = \chi_{td}(C_{2n})$ . □

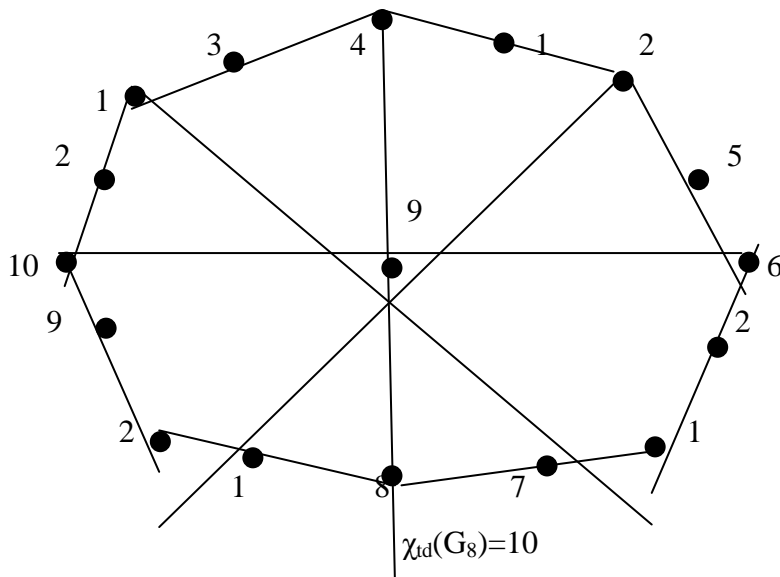
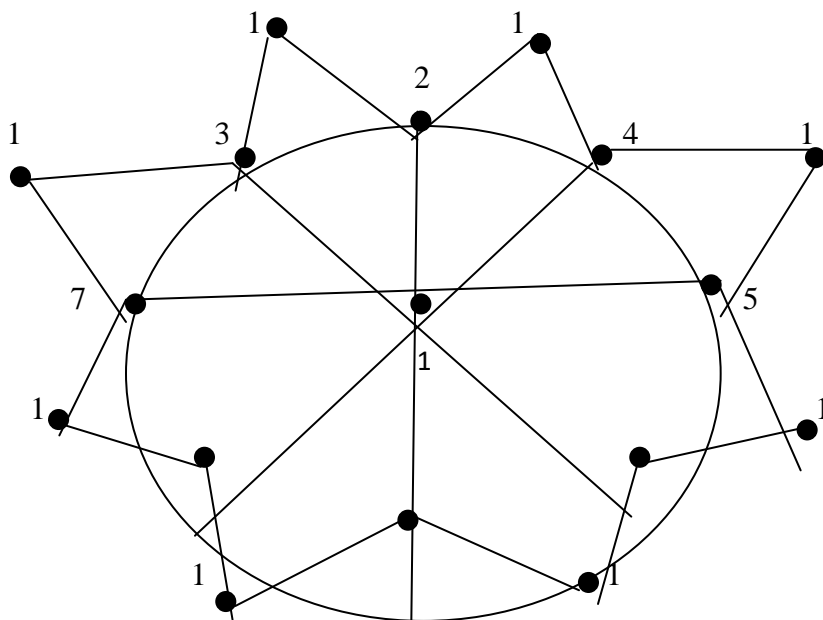


Fig 6. Gear Graph

**Theorem 7** Any sunflower graph  $Sf_n$ ,  $\chi_{td}(Sf_n) = 1 + \chi_{td}(C_n)$

**Proof:** Let  $Sf_n$  be the sun flower graph obtained taking a wheel with central vertex  $v_0$  and the cycle  $C_n$  ( $v_1v_2v_3\text{-----}v_n v_1$ ) and new vertices  $w_1, w_2, w_3, \text{----}, w_n$  where  $w_i$  is joined by the vertices  $v_i, v_{i+1}$ . We assign the color 1 to the set of vertices  $\{v_0, w_1, w_2, w_3, \text{----}, w_n\}$ . Remaining vertices lies on the circle  $C_n$ , for td-coloring, we need  $\chi_{td}(C_n)$  colors. Thus  $\chi_{td}(Sf_n) = 1 + \chi_{td}(C_n)$ . □



**Fig 7 Sun Flower Graph**

$$\chi_{td}(Sf_8) = 7$$

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