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Eigenvalues Estimation For Quaternion Symmetric Positive Semide finite Matices

Rajeswari J[1] and Gunasekaran. K[2*]

[1]Research Scholar, Government Arts College(Autonomous), Kumbakonam, Tamilnadu, India.rajeswari1983bose@gmail.com

[2]Associate Professor and Head, Department of Mathematics, Government Arts College(Autonomous), Kumbakonam, Tamilnadu, India, drkgsmath@gmail.com

ABSTRACT

We use the fact that the set of quaternion symmetric positive semi definite matrices of order n form a cone with a special structure, in order to find bounds for the quaternion eigenvalues of a quaternion symmetric matrix.

KEYWORDS: Quaternion symmetric; Quaternion symmetric positive definite matrices; Quaternion eigenvalues; Fresenius norm; Quaternion hermitian matrices.

***Corresponding Author**

Dr. GunasekaranK

Associate Professor and Head,

Department of Mathematics,

Government Arts College(Autonomous),

Kumbakonam – 612 002 Tamilnadu, India.

Mob: 9442549689 Email: drkgsmath@gmail.com

1. INTRODUCTION

Quaternion symmetric and quaternion symmetric positive definite matrices have been extensively studied and there are good characterizations of these sets. We wish to use the fact that in $\mathrm{H}_{n\times n}$, the set of quaternion symmetric positive semidefinite matrices forms a cone with a very special structure: the identity matrix is the central direction and there exist certain kinds of symmetries around it. The position of each matrix in the cone depends strongly on its quaternion eigenvalues and consequently on its rank. We exploit this special structure below.

First, we observe that, when the rank of a quaternion symmetric positive semidefinite matrix decreases, then its angle with the identity matrix increases. In this sense,the rank one matrices are the farthest from the identity and all of them form a fixed angle with that matrix.

In the final section, the following bounds for the quaternion eigenvalues of any quaternion symmetric matrix[4] $X_0 + X_1 j$ are obtained. If $\lambda_1, \lambda_2 .....\lambda_n$ are the quaternion eigenvalues of A and we denote Frobenius norm by $\|.\|_F$ respectively, we prove that

$$\left|\frac{trace\left(X_0+X_1 j\right)}{n}-\lambda_s\right| \leq \left[\left(\frac{n-1}{n}\right)\left\|X_0+X_1 j\right\|_F^2-\frac{trace\left(X_0+X_1 j\right)^2}{n}\right]^{1/2}$$

for λ_s, $s = 1, 2.....n$.

Finally, one more relation is established for quaternion symmetric positive semidefinite matrices[1,2] and it concerns the number of quaternion eigenvalus above their mean. We show that when the angle between any matrix and the identity increases, then the number of quaternion eigenvalues above the mean decreases in the following manner have been proved: at most $t-1$ quaternion eigenvalues[6] are above the mean if $trace\left(X_0+X_1 j\right)/\left\|X_0+X_1 j\right\|_F < p_t^{-1}$, where p_t is defined by

$$p_t(\alpha)^{-1} = \left(\frac{\alpha-t+1}{\alpha}\right)^2 + (t-1)(\tfrac{1}{\alpha})^2$$

External examples of this relation are the identity, withall its quaternion eigenvalues[3] at the mean, and the rank one matrix, with only one quaternion eigenvalue above the mean. This relation is valid not only for the mean but for any number in the interval $\left[0, trace(X_0 + X_1 j)\right]$: then similar results can be obtained.

2. NOTATION AND FIRST RESULTS

The set of quaternion hermitian matrices of order n isdenote by $\mathrm{H}_{n\times n}$, and by Ω_n the matrices in $\mathrm{H}_{n\times n}$ that are quaternion positive semidefinite[4]. Now for $t = 1, 2, ...n$, we can define the following subsets:

$$\Omega_t^{=} = \left\{ X_0 + X_1 j \in \Omega_n \,/\, rank(X) = t \right\} \text{ and}$$

$\Omega_t = \left\{ X_0 + X_1 j \in \Omega_n \,/\, rank(X) \le t \right\}$

[where $X = X_0 + X_1 j$, $rank(X) = rank(X_0 + X_1 j) \le rank(X_0) + rank(X_1) j$]We will use in $H_{n\times n}$ the set of quarternionhermitiansquare matrices of order n, the Frobenius inner product defined by

$$\left\langle X_0 + X_1 j, Y_0 + Y_1 j \right\rangle_F = \left| trace(X^{CT} Y) \right|$$

$$= \left| trace\left[(X_0^{CT} - X_1^{CT} j)(Y_0 + Y_1 j) \right] \right|$$

$$= \left| trace(X_0^{CT} Y_0) - trace(X_1^{CT} Y_1 j) \right|$$

This inner product allows us to define the cosine of the angle between two quaternion hermitian matrices in $H_{n\times n}$ by

$$\cos\left(X_0 + X_1 j, Y_0 + Y_1 j \right) = \frac{\left\langle X_0 + X_1 j, Y_0 + X_1 j \right\rangle_F}{\left\| X_0 + X_1 j \right\|_F \left\| Y_0 + Y_1 j \right\|_F}$$

For the 1-norm and 2-norm in $H_{n\times n}$ we will use the usual notation. It is well known that Ω_n is a cone and its interior is the set of quaternion hermitian positive definite matrices. We show the location of the sets Ω_t, $t \ne n$ in Ω_n with respect to the identity matrix, which will be noted by I. The following simple result can be had.

Lemma 2.1

If $X = X_0 + X_1 j \in \Omega_t^{=}$, then $trace\left(X_0 + X_1 j \right) / \left\| X_0 + X_1 j \right\|_F \le t^{1/2}$,

Proof

Since $X = X_0 + X_1 j \in \Omega_t^{=}$, $X_0 + X_1 j$ has t, positive eigenvalues $\lambda_1, \lambda_2, ...\lambda_t$. We denote by λ ,the vector with those components. We need to compute the cosine of the angle formed between $X_0 + X_1 j$ and I as.

$$\cos\left(X_0 + X_1 j, I \right) = \frac{\left\langle X_0 + X_1 j, I \right\rangle_F}{\left\| X_0 + X_1 j \right\|_F \left\| I \right\|_F} = \frac{trace\left(X_0 + X_1 j \right)}{\left\| X_0 + X_1 j \right\|_F n^{1/2}}$$

[$\because$ Regarding the Frobenius norm $\left\| I_n \right\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij}^2} = \sqrt{n}$ where δ_{ij} is the Kronecker symbol, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$]

But using the facts $trace(X_0) = \left\| \lambda_1 \right\|_1$, $trace(X_1 j) = \left\| \lambda_2 \right\|_1$, $\left\| X_0 \right\| = \left\| \lambda_1 \right\|_2$,and $\left\| X_1 \right\| = \left\| \lambda_2 \right\|_2$ the inequality between 1-norm and 2-norm, we have

$$\frac{trace(X_0+X_1j)}{\|X_0+X_1j\|_F\, n^{1/2}} \le \frac{trace(X_0)+trace(X_1j)}{\|X_0\|_F+\|X_1j\|_F\, n^{1/2}}$$

$$\le \frac{\|\lambda_1\|_1+\|\lambda_2\|_1}{\|\lambda_1\|_2+\|\lambda_2\|_2\, n^{1/2}}$$

$$\frac{trace(X_0+X_1j)}{\|X_0+X_1j\|_F\, n^{1/2}} \le \frac{k^{1/2}}{n^{1/2}}$$

Which proves the lemma.

Remark 2.2

We want to make some observations: First, the lemma is valid for $X = X_0 + X_1 j \in \Omega_t$; second, the contra positive statement gives us a lower bound for the rank of $(X_0 + X_1 j)$; and finally, if $X_0 + X_1 j \in \Omega_t^=$ has all equal quaternion eigenvalues, then equality holds. Any element in $\Omega_t^=$ can be written as

$$\sum_{s=1}^{t}(a_0+a_1j)_s\,(a_0+a_1j)_s^{CT} = \sum_{s=1}^{t}\left(a_{0(s)}a_{0(s)}^{CT} - a_{1(s)}a_{1(s)}^{CT}j\right)$$

for some $a_s \in \mathrm{H}_{n\times n}, s = 1,2,.....t$ (see [1,2]). In particular, any element in Ω_1 takes the form $d_0d_0^T + d_1d_1^T j$ for $d_0 + d_1 j \in \mathrm{H}_{n\times n}$. We are interested now in the projection of any element of $(D_0 + D_1 j)_n$ on the direction generated by elements of Ω_1.

Lemma 2.3

Let $d_0d_0^T + d_1d_1^T j$ be an element of Ω_1. The projection of $X_0 + X_1 j \in (D_0 + D_1 j)_n$ on the $d_0d_0^T + d_1d_1^T j$ is

$$\frac{d_0^T X_0 d_0 - d_1^T X_1 d_1 j}{d_0^T d_0 + d_1^T d_1 j}.\frac{d_0 d_0^T + d_1 d_1^T j}{d_0^T d_0 + d_1^T d_1 j}$$

Proof

The desired projection is given by

$$\|X_0+X_1j\|_F \cos\left(X_0+X_1j, d_0d_0^T+d_1d_1^T j\right).\frac{d_0d_0^T+d_1d_1^T j}{d_0^T d_0 + d_1^T d_1 j}$$

we can compute now the coefficient of the quaternion unitary matrix $\frac{d_0d_0^T+d_1d_1^T j}{d_0^T d_0 + d_1^T d_1 j}$ (quaternion unitary matrix means in this article a matrix with Frobenius norm equal to one):

$$\left\|X_0+X_1 j\right\|_F \cos\left(X_0+X_1 j, d_0 d_0^T+d_1 d_1^T j\right) \cdot \frac{d_0 d_0^T+d_1 d_1^T j}{d_0^T d_0+d_1^T d_1 j}$$

$$=\left\|X_0+X_1 j\right\|_F \frac{\left\langle X_0+X_1 j, d_0 d_0^T+d_1 d_1^T j\right\rangle_F}{\left\|X_0+X_1 j\right\|_F\left\|d_0 d_0^T+d_1 d_1^T j\right\|_F}\left[\because \cos(A, B)=\frac{\langle A, B\rangle_F}{\|A\|_F\|B\|_F}\right]$$

$$=\frac{trace\left[\left(X_0^{CT}-X_1^{CT} j\right)\left(d_0 d_0^T+d_1 d_1^T j\right)\right]}{\left\|d_0 d_0^T+d_1 d_1^T j\right\|_F}$$

$$\left[\because\left\langle X_0+X_1 j, d_0 d_0^T+d_1 d_1^T j\right\rangle_F=trace\left[\left(X_0^{CT}-X_1^{CT} j\right)\left(d_0 d_0^T+d_1 d_1^T j\right)\right]\right]$$

$$=\frac{d_0^T X_0 d_0-d_1^T X_1 d_1 j}{d_0 d_0^T+d_1 d_1^T j}$$

Note 2.4

It is very important to observe that the coefficient computed is the Raylaigh quotient.

3. MAIN RESULTS

We consider now the following set:

$\gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)=\left\{\left(M_0+M_1 j\right) \in \mathrm{H}_{n\times n} \left\|\cos\left(M_0+M_1 j, \mathrm{I}\right)\right\|=\frac{1}{n^{1/2}}\right\}$. For $n=2$ is easy to prove that

$\Omega_1=\gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)$ but for $n>2$ we get $\Omega_1 \subset \gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)$.

Using $\gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)$ allows us to get lower and upper bounds for the quaternion eigenvalues that are invariant under similar quaternion orthogonal transformations. Using this invariance then allows us to write the trace and its Frobenius norm in terms of quaternion eigenvalues. Thus it is easy to see that

$$\left[\frac{1}{2}\left(\left\|X_0+X_1 j\right\|_F^2-\frac{trace\left(X_0+X_1 j\right)^2}{2}\right)\right]^{1/2}$$

is the standard derivation of the quaternion eigenvalues λ_s, $s=1,2,\ldots..n$ this can be computed even if the quaternion eigenvalues are unknown. We denote the standard deviation by $d(\lambda)$.

Theorem 3.1

Let $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ be the quaternion eigenvalues of $X_0+X_1 j \in\left(D_0+D_1 j\right)_n$. Then each λ_s, $s=1,2,\ldots..n$ satisfies

$$\left|\lambda_s - \frac{trace\left(X_0 + X_1 j\right)}{n}\right| \le \left[\frac{n-1}{n}\left(\left\|X_0 + X_1 j\right\|_F^2 - \frac{trace\left(X_0 + X_1 j\right)^2}{2}\right)\right]^{1/2}$$

Proof

If λ_n is the maximum quaternion eigenvalues of $X_0 + X_1 j$, then by the properties of the Rayleigh quotient we have

$$\begin{aligned}
\lambda_n &= \max_{(d_0 + d_1 j) \in \mathrm{H}_{n\times n}} \frac{d_0^T X_0 d_0 - d_1^T X_1 d_1 j}{d_0 d_0^T + d_1 d_1^T j} \\
&= \max_{(d_0 + d_1 j) \in \mathrm{H}_{n\times n}} \left\|X_0 + X_1 j\right\|_F \cos\left(X_0 + X_1 j, d_0 d_0^T + d_1 d_1^T j\right) \\
&= \max_{(M_0 + M_1 j) \in \gamma\left(I, \frac{1}{n^{1/2}}\right)} \left\|X_0 + X_1 j\right\|_F \cos\left(X_0 + X_1 j, M_0 + M_1 j\right)
\end{aligned}$$

where the last inequality is consequence of the inclusion $\Omega_1 \subset \gamma\left(I, \frac{1}{n^{1/2}}\right)$.

Our goal is to compute $\max_{(M_0 + M_1 j) \in \gamma\left(I, \frac{1}{n^{1/2}}\right)} \left\|X_0 + X_1 j\right\|_F \cos\left(X_0 + X_1 j, M_0 + M_1 j\right)$. In order to do that, we transform our function as follows:

$$\begin{aligned}
\left\|X_0 + X_1 j\right\|_F \cos\left(X_0 + X_1 j, M_0 + M_1 j\right) &= \left\|X_0 + X_1 j\right\|_F \frac{\left\langle X_0 + X_1 j, M_0 + M_1 j\right\rangle_F}{\left\|X_0 + X_1 j\right\|_F \left\|M_0 + M_1 j\right\|_F} \\
&= \frac{\left\langle X_0 + X_1 j, M_0 + M_1 j\right\rangle_F}{\left\|M_0 + M_1 j\right\|_F}
\end{aligned}$$

We need now an appropriate parametrizationof the vectors of $\gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)$. We propose the following expression for $M_0 + M_1 j \in \gamma\left(\mathrm{I}, \frac{1}{n^{1/2}}\right)$:

$$M_0 + M_1 j = \frac{1}{n^{1/2}} \frac{I}{n^{1/2}} + \sigma_1 \frac{X_0^{CT} - X_1^{CT} j}{\left\|X_0^{CT} - X_1^{CT} j\right\|_F} + \sigma_2 \frac{Y_0 + Y_1 j}{\left\|Y_0 + Y_1 j\right\|_F}$$

In this expression $X_0^{CT} - X_1^{CT} j$ is the projection of $X_0 + X_1 j$ on the subspace orthogonal to I and $Y_0 + Y_1 j$ is orthogonal to I and $X_0 + X_1 j$. In order to get $M_0 + M_1 j$ quaternion unitary, we require that $\sigma_1^2 + \sigma_2^2 = \frac{n-1}{n}$. Now, we are ready to compute

$$\frac{\left\langle X_0+X_1j, M_0+M_1j\right\rangle}{\left\|M_0+M_1j\right\|_F} =$$

$$\frac{1}{n^{1/2}}\frac{\left\langle X_0+X_1j, I\right\rangle_F}{n^{1/2}}+\sigma_1\left\langle X_0+X_1j, \frac{X_0^{CT}-X_1^{CT}j}{\left\|X_0^{CT}-X_1^{CT}j\right\|_F}\right\rangle_F+\sigma_2\left\langle X_0+X_1j, \frac{Y_0+Y_1j}{\left\|Y_0+Y_1j\right\|_F}\right\rangle_F$$

$$= \frac{1}{n}trace\left(X_0+X_1j\right)+\sigma_1\left(\left\|X_0+X_1j\right\|_F^2-\frac{trace\left(X_0+X_1j\right)^2}{n}\right)^{1/2}$$

and is easy to see that

$$\max_{(M_0+M_1j)\in\gamma\left(I,\frac{1}{n^{1/2}}\right)}\left\|X_0+X_1j\right\|_F\cos\left(X_0+X_1j, M_0+M_1j\right)=\max_{\substack{(M_0+M_1j)\in\gamma\left(I,\frac{1}{n^{1/2}}\right)\\ \left\|M_0+M_1j\right\|_F=1}}\left\langle X_0+X_1j, M_0+M_1j\right\rangle_F$$

$$=\frac{1}{n}trace\left(X_0+X_1j\right)+\left(\frac{n-1}{n}\right)^{1/2}\left(\left\|X_0+X_1j\right\|_F^2-\frac{trace\left(X_0+X_1j\right)^2}{n}\right)^{1/2}$$

Because $\left(\left\|X_0+X_1j\right\|_F^2-\frac{trace\left(X_0+X_1j\right)^2}{n}\right)^{1/2}\geq 0$

The lower bound can be computed in a very similar way.

Remark 3.2

For $n=2$, these bounds are exactly the quaternion eigenvalues.

Remark 3.3

In the inequality of the above theorem, equality holds for $X_0+X_1j\in\Omega_1$, or when X_0+X_1j is a multiple of the identity matrix.

Another interesting thing to know is if there exist quaternion eigenvalues in the intervals

$$\left[\frac{trace\left(X_0+X_1j\right)}{n}-(n-1)^{1/2}\left(d_0+d_1j\right)(\lambda), \frac{trace\left(X_0+X_1j\right)}{n}-\left(d_0+d_1j\right)(\lambda)\right] \text{and}$$

$$\left[\frac{trace\left(X_0+X_1j\right)}{n}-\left(d_0+d_1j\right)(\lambda), \frac{trace\left(X_0+X_1j\right)}{n}-(n-1)^{1/2}\left(d_0+d_1j\right)(\lambda)\right]$$

the following result concerns this quaternion.

Remark 3.4

At least one of the maximum and the minimum quaternion eigenvalue is in one of the intervals mentioned just above.

Proof

It is a consequence of the fact that the standard deviation is less than or equal to the absolute value of the maximum deviation.

Remark 3.5

Another question that we want to answer for matrices in Ω_n ,is, how are the eigenvalues located with respect to their mean? There exists a relation between the angle that the matrix $X_0 + X_1 j$,forms with the identity matrix and the number of quaternion eigenvalues above the mean. The following result establishes this relation. We need to introduce some special values p_t for $t = 1, 2......n$ defined by $p_t^{-1} = \left(\frac{n-t+1}{n}\right)^2 + (t-1)\left(\frac{1}{2}\right)^2$

Theorem 3.6

If $X_0 + X_1 j \in \Omega_n$ has only quaternion eigenvalues greater or equal to $\frac{trace(X_0 + X_1 j)}{n}$ then $\frac{trace(X_0 + X_1 j)}{\|X_0 + X_1 j\|_F} \geq p_t^{1/2}$.

Proof

We can assume that $trace(X_0 + X_1 j) = 1$, because this does not affect either the angle between $X_0 + X_1 j$ and I or the order relation between quaternion eigenvalues and its means. We recall that

$$\cos(X_0 + X_1 j, I) = \frac{trace(X_0 + X_1 j)}{\|\lambda_0 + \lambda_1 j\|_2 n^{1/2}}.$$

Then it is only necessary to prove that $\frac{1}{\|\lambda_0 + \lambda_1 j\|} \geq p_t^{1/2}$ or $p_t^{-1} \geq \sum_{s=1}^{n} \left(\lambda_{0(s)} + \lambda_{1(s)} j\right)^2$

But this is clear, because p_t^{-1} is the optimum for the problem

$$\max g(\lambda_1, \lambda_2 .....\lambda_n) = \sum_{s=1}^{n} \left(\lambda_{0(s)} + \lambda_{1(s)} j\right)^2$$

Such that $\sum_{s=1}^{n} \left(\lambda_{0(s)} + \lambda_{1(s)} j\right) = 1$

$$(\lambda_0 + \lambda_1 j)_{n-s} \geq \tfrac{1}{n} \; s = 0, 1, 2, ...(s-1)$$

$$(\lambda_0 + \lambda_1 j)_s \geq 0 \; s = 0, 1, 2, ...n$$

An easy proof of the last statement can be given using the Kuhn-Tucker conditions.

Note 3.7

The useful version of this result is the following.

Remark 3.8

Let $X_0 + X_1 j \in \Omega_n$. If $\dfrac{trace(X_0 + X_1 j)}{\|X_0 + X_1 j\|} < p_t^{1/2}$ then at most $t-1$.quaternion eigenvalues are above $\dfrac{trace(X_0 + X_1 j)}{n}$.

Remark 3.9

If $X_0 + X_1 j \in \Omega_n$ and $\dfrac{trace(X_0 + X_1 j)}{\|X_0 + X_1 j\|_F} \geq p_t^{1/2}$, then $X_0 + X_1 j$ has only one quaternion eigenvalue above $\dfrac{trace(X_0 + X_1 j)}{n}$, and it belongs to the interval

$$\left[\frac{trace(X_0 + X_1 j)}{n} + (d_0 + d_1 j)(\lambda_0 + \lambda_1 j), \frac{trace(X_0 + X_1 j)}{n} + (n-1)^{1/2}(d_0 + d_1 j)(\lambda_0 + \lambda_1 j)\right]$$

Finally we want to note that similar results to Theorem 3.7 and remark 3.8 can be obtained with identical proofs for every α in the interval $\left[0, trace(X_0 + X_1 j)\right]$. The values p_t are now defined by

$$p_t(\alpha)^{-1} = \left(\frac{\alpha - t + 1}{\alpha}\right)^2 + (t-1)(\tfrac{1}{\alpha})^2$$

All results formulated forsymmetric positive semidefinite matrices have corresponding ones for quaternion symmetric negative semidefinite matrices.

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