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Expansions Formula Involving G-Function of Two Variables

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ABSTRACT:

The G-function, a generalization of the hyper geometric function, can be defined by a Mellin-Barnes contour integral or represented as a sum of hyper geometric functions. The expansions of Meijer's G-functions in a series of similar functions and their products with terminating hyper geometric functions, have been studied by several mathematicians.

Recently, various authors have evaluated certain number of expansion formulae involving generalized hyper geometric functions. These expansions can be written out easily from the known expansions of elementary functions by using induction through Laplace transform and its inverse. However, it is strange to notice that there is not even a single known expansion of Meijer's G-function in a series of product of G-functions.

Looking into the requirement and importance of various properties of expansion in several field, in this paper we have obtained some expansion formula for G-Function of two variables involving Bessel functions with the help of the orthogonality property of Bessel functions. The functions arise as solutions of the Laplace, wave, heat, Helmholtz, and Schrooinger equations, and new bases can be constructed from the functions with which to expand general solutions of these physically important equations.

KEY WORDS: G-Function of two variables, expansion formula.

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1. INTRODUCTION:

Recently, Samtani[5], Agrawal[1], Goyal[2] and others have evaluated certain number of expansion formulae involving generalized hypergeometric functions. The G-function of two variables was defined by Shrivastava and Joshi[6] in terms of Mellin-Barnes type integrals as follows:

$$G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}\\(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}\end{matrix}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\xi+\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\xi-\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\xi+\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, $(i = 1, 2, 3; j = 2, 3)$.

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi)$ $(j = 1, ..., m_2)$ lie to the right, and the poles of $\Gamma(1 - c_j + \xi)$ $(j = 1, ..., n_2)$, $\Gamma(1 - a_j + \xi + \eta)$ $(j = 1, ..., n_1)$ to the left of the contour.

The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta)$ $(j = 1, ..., m_3)$ lie to the right, and the poles of $\Gamma(1 - e_j + \eta)$ $(j = 1, ..., n_3)$, $\Gamma(1 - a_j + \xi + \eta)$ $(j = 1, ..., n_1)$ to the left of the contour, and the double integral converges if

$$2(n_1 + m_2 + n_2) > (p_1 + q_1 + p_2 + q_2)$$

$$2(n_1 + m_3 + n_3) > (p_1 + q_1 + p_3 + q_3)$$

and $\qquad |\arg x| < \tfrac{1}{2}U\pi, \; |\arg y| < \tfrac{1}{2}V\pi.$

where $\qquad U = [n_1 + m_2 + n_2 - \tfrac{1}{2}(p_1 + q_1 + p_2 + q_2)]$

$$V = [n_1 + m_3 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_3 + q_3)]$$

These assumptions for the G-function of two variables will be adhered to throughout this research work.

The following formulae are required in the proof:

From Luke[3]:

$$\int_0^{\infty} e^{ix}\, x^{\mu-1} J_\nu(x) = \frac{e^{\frac{1}{2}i(\mu+\nu+1)\pi}\Gamma(\mu+\nu+1)\Gamma(-\mu-\frac{1}{2})}{2^{\mu+1}\Gamma(\frac{1}{2})\Gamma(\nu-\mu)}, \tag{2}$$

where $\mathrm{Re}(\mu+\nu) > -1, Re\ \mu < -\frac{1}{2}$.

$$\int_0^{\infty} t^{-1}\, J_{\nu+2n+1}(t) J_{\nu+2m+1}(t) dt = \begin{cases} 0, \text{if } m \neq n \\ (4n+2\nu+2)^{-1}, \text{if } m = n, \end{cases} \tag{3}$$

$$\mathrm{Re}(\nu) + m + n > -1$$

From Rainville[4], Legendre's duplication formula:

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2Z-1}\Gamma(z)\, \Gamma(z + ½). \tag{4}$$

2. INTEGRAL:

In this section, we shall establish the following Integral:

$$\int_0^{\infty} e^{ix}\, x^{\mu-1} J_\nu(x) G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix} z_1x^2 \\ z_2 \end{matrix}\middle| \begin{matrix} (a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3} \\ (b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3} \end{matrix}\right]dx$$

$$= \frac{e^{\frac{1}{2}i(\mu+\nu)\pi}}{2^{3/2}\,\pi^2} G^{0,n_1:m_2+2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+4,q_2+2;p_3,q_3}\left[\begin{matrix} z_1e^{\pi i} \\ z_2 \end{matrix}\right.$$

$$\left|\begin{matrix} (a_j;1,1)_{1,p_1}:\left(\frac{2-\mu-\nu}{2},1\right),\left(\frac{1-\mu-\nu}{2},1\right)(c_j,1)_{1,p_2},\left(\frac{\nu-\mu+1}{2},1\right),\left(\frac{\nu-\mu+2}{2},1\right):(e_j,1)_{1,p_3} \\ (b_j;1,1)_{1,q_1}:\left(\frac{-\mu+\frac{1}{2}}{2},1\right),\left(\frac{-\mu+\frac{3}{2}}{2},1\right),(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3} \end{matrix}\right] \tag{5}$$

where $0 \le \theta \le \pi$, $2(m_2+n_2) > p_2 + q_2$, $|\arg z_1| < (m_2 + n_2 - \frac{p_2}{2} - \frac{q_2}{2})\pi$,

$\mathrm{Re}(\mu+\nu+2d_j) > -1, j = 1,2, \dots, m_2, \mathrm{Re}\ \mu < -1/2.$

Proof of (5):

To establish (5), expressing the G-function of two variables on the left-hand side as Mellin-Barnes type integral (1), we have

$$\int_0^{\infty} e^{ix}\, x^{\mu-1} J_\nu(x)\left[\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)](z_1x^2)^{\xi}(z_2)^{\eta}d\xi d\eta\right]dx$$

On interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we have

$$\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\left[\int_0^{\infty} e^{ix}\, x^{\mu+2\xi-1} J_\nu(x)dx\right](z_1)^{\xi}(z_2)^{\eta}d\xi d\eta.$$

Now evaluating the inner integral with the help of formula (2) and using Legendre's duplication formula (4), we have

$$\frac{e^{\frac{1}{2}i(\mu+\nu)\pi}}{2^{3/2}\,\pi^2}\cdot\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta) \times$$

$$\times \frac{\Gamma(\frac{\mu+\nu}{2}+\xi)\Gamma(\frac{\mu+\nu+1}{2}+\xi)\Gamma(\frac{-\mu+\frac{1}{2}}{2}-\xi)\Gamma(\frac{-\mu+\frac{3}{2}}{2}-\xi)}{\Gamma(\frac{\nu-\mu+1}{2}-\xi)\Gamma(\frac{\nu-\mu+2}{2}-\xi)}\left(z_1e^{\pi i}\right)^{\xi}(z_2)^{\eta}d\xi d\eta.$$

On applying (1), the integral (5) is established.

3. EXPANSION:

The expansion formula to be established is

$$e^{ix}x^{\mu}G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{z_1x^2 \atop z_2}\Big|{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3} \atop (b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right]$$

$$=\sum_{s=0}^{\infty}\frac{e^{\frac{1}{2}i(\mu+\nu)\pi}}{2^{1/2}\,\pi^2}\nu J_{\nu}(x)G^{0,n_1:m_2+2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+4,q_2+2;p_3,q_3}\left[{z_1e^{\pi i} \atop z_2}\right.$$

$$\left|{(a_j;1,1)_{1,p_1}:\left(\frac{2-\mu-\nu}{2},1\right),\left(\frac{1-\mu-\nu}{2},1\right)(c_j,1)_{1,p_2},\left(\frac{\nu-\mu+1}{2},1\right),\left(\frac{\nu-\mu+2}{2},1\right):(e_j,1)_{1,p_3} \atop (b_j;1,1)_{1,q_1}:\left(\frac{-\mu+\frac{1}{2}}{2},1\right),\left(\frac{-\mu+\frac{3}{2}}{2},1\right),(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right], \quad (6)$$

where $2(m_2+n_2)>p_2+q_2$, $|\arg z_1|<(m_2+n_2-\frac{p_2}{2}-\frac{q_2}{2})\pi$,

$\mathrm{Re}(\mu+2d_j)>0, j=1,2,\ldots,m_2, \mathrm{Re}\,\mu<\frac{1}{2}$; $\nu=u+2s+1$.

Proof of (6):

Let $f(x)=e^{ix}x^{\mu}G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{z_1x^2 \atop z_2}\Big|{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3} \atop (b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right]$

$$=\sum_{s=0}^{\infty}C_s\,J_{u+2s+1}(x), 0<x<\infty. \quad (7)$$

Equation (7) is valid since f(x) is continuous and of bounded variation in the open interval (0, ∞) when μ ≥ 0. Now multiplying both sides of (7) by $x^{-1}J_{u+2t+1}(x)$ and integrating from 0 to ∞, with respect to x, we get

$$\int_0^{\infty}e^{ix}x^{\mu-1}J_{u+2t+1}(x)G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{z_1x^2 \atop z_2}\Big|{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3} \atop (b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right]dx$$

$$=\sum_{s=0}^{\infty}C_s\int_0^{\infty}x^{-1}J_{u+2t+1}(x)J_{u+2s+1}(x)dx.$$

Now using (5) and the orthogonality property of Bessel functions (3), we have

$$C_t=\frac{e^{\frac{1}{2}i(\mu+\nu)\pi}}{2^{1/2}\,\pi^2}\nu G^{0,n_1:m_2+2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+4,q_2+2;p_3,q_3}\left[{z_1e^{\pi i} \atop z_2}\right.$$

$$\left|{(a_j;1,1)_{1,p_1}:\left(\frac{2-\mu-\nu}{2},1\right),\left(\frac{1-\mu-\nu}{2},1\right)(c_j,1)_{1,p_2},\left(\frac{\nu-\mu+1}{2},1\right),\left(\frac{\nu-\mu+2}{2},1\right):(e_j,1)_{1,p_3} \atop (b_j;1,1)_{1,q_1}:\left(\frac{-\mu+\frac{1}{2}}{2},1\right),\left(\frac{-\mu+\frac{3}{2}}{2},1\right),(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right], \quad (8)$$

where $\nu=u+2t+1$.

From (7) and (8) the formula (6) is obtained.

4. SPECIAL CASES:

On specializing the parameters in (6), we get following expansion formula in terms of G-function of one variable:

$$e^{ix}x^{\mu}G_{p,q}^{m,n}\left[zx^2\middle|{}^{(a_j,1)_{1,p}}_{(b_j,1)_{1,q}}\right]$$

$$= \sum_{s=0}^{\infty} \frac{e^{\frac{1}{2}i(\mu+\nu)\pi}}{2^{1/2}\,\pi^2} \nu J_{\nu}(x) \times$$

$$\times\, G_{p+4,q+2}^{m+2,n+2}\left[ze^{\pi i}\middle|{}^{\left(\frac{2-\mu-\nu}{2},1\right),\left(\frac{1-\mu-\nu}{2},1\right),(a_j,1)_{1,p},\left(\frac{\nu-\mu+1}{2},1\right),\left(\frac{\nu-\mu+2}{2},1\right)}_{\left(\frac{-\mu+\frac{1}{2}}{2},1\right),\left(\frac{-\mu+\frac{3}{2}}{2},1\right),(b_j,1)_{1,q}}\right] \qquad (9)$$

where $2(m+n) > p+q$, $|\arg z| < (m+n-\frac{p}{2}-\frac{q}{2})\pi$,

$\mathrm{Re}(\mu+2b_j) > 0, j = 1,2,\ldots,m, Re\,\mu < \frac{1}{2};\ \nu = u+2s+1.$

5. CONCLUSION:

On specializing the parameters, G-function may be reduced to several other higher transcendental functions. Therefore the result (6) and (9) are of general nature and may reduce to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

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