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An Alexandroff Bitopological Space on Undirected Graphs

D. Sasikala* and Divya. A[1]

*Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore-641004, Tamilnadu, India. E-mail:dsasikala@psgrkcw.ac.in
[1]Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore-641004, Tamilnadu, India. E-mail: divya772248@gmail.com

ABSTRACT

All nodes (vertices) has finitely many adjacent nodes(vertices) in graph $G = (V, E)$ is called locally finite graph, where V is referred as vertex (node) set and E is referred as edge (arc) set. Alexandroff spaces became much more important field because of their use in digital topology. Alexandroff space is a topological space, in which arbitrary intersection of open sets is open (or arbitrary union of closed sets is closed) equivalently, we say that each singleton has minimal neighborhood base. The bitopological spaces that is the triple (A, τ_1, τ_2) of a collection A with two (arbitrary) topologies τ_1 and τ_2 on A. In this paper, we mean by a bitopological space (V, τ_G,τ_{IG}) is an Alexandroff bitopological space, satisfy the stronger condition namely, arbitrary intersection of members of S_G and S_{IG} are open in τ_G and τ_{IG} respectively on V, where S_G is the sub basis for a graphic topology τ_G and S_{IG} is the sub basis for a incident topology τ_{IG}. Latter, we investigate some properties and characterization of this topological spaces. In particular, the separation axioms are studied. Our goal is to consider the fundamental steps toward analyzing some properties of locally finite graphs by their corresponding topology.

KEYWORDS : Locally finite graphs, Alexandroff space, bitopological space.

*Corresponding Author

Dr. D. Sasikala*

Department of Mathematics,

PSGR Krishnammal College for Women, Coimbatore-641004,

Tamilnadu, India.

E-mail:dsasikala@psgrkcw.ac.in., Mob.No.-9976806838

1 INTRODUCTION:

Now a days the graphs structure had heaps of applications in our real life. Mathematical structure of the graphs are helpful from practical perspective once they abstractly represents by graphs[9]. Collection of nodes may be connected in graphs is employed nowadays to review issues in economics, networks of communication, knowledge organization, procedure devices, physics, chemistry, sociology, linguistics and numberless alternative fields. Topology is that the one in every of the foremost necessary fields in mathematics. A set becomes a topological space. It is the geometry part that does not concern distance. The one topology is extended to bitopology with usual definition. Kelly[8] was the first who developed the ideas of bitopological spaces that is the triple (A, τ_1, τ_2) of a collection A with two (arbitrary) topologies τ_1 and τ_2 on A. Baby Girija and Pilakkat[2] introduced bitopological spaces associated with digraphs *D = (V, E)* employing a nonnegative real valued function P on VXV known as quasi pseudo metric that develops two topologies on V. And then Khalid Abdulkalek Abdu and Adem kilicman introduced Bitopological spaces on undirected graphs *G = (V, E)* by using two different sub basis families to come up with two topologies on V[9].

In this paper we tend to think about solely locally finite simple graphs to this Alexandroff bitopological space.

2 PRELIMINARIES*:*

The bitopological space (A, τ_1, τ_2) is pair wise T_0 if for each pair (u,v) of distinct points of A there is either a τ_1-open set containing u but not v or there exists a τ_2-open set containing v but not u. If for each pair of distinct points there exists a τ_1-open set or τ_2-open set containing one but the other, then (A, τ_1, τ_2) is weakly pair wise T_0. If for each pair of distinct points u, v there exists a τ_1-open set D and a τ_2-open set W such that $u \in D$, $v \notin D$ and $v \in W$, $u \notin W$ or $u \in W$, $v \notin W$ and $v \in D$, $u \notin D$ then (A, τ_1, τ_2) is weakly pair wise T_1. The bitopological space (A, τ_1, τ_2) is weakly pair wise T_2 if for each pair of distinct points u, v there exists a τ_1-open set D and a τ_2-open set W with $D \cap W = \emptyset$ such that $u \in D$ and $v \in W$ or $u \in W$ and $v \in D$. If for each pair of distinct points (u,v) there exists a τ_1-open set D and a τ_2-open set W with $D \cap W = \emptyset$ such that $u \in D$ and $v \in W$ then (A, τ_1, τ_2) is pair wise T_2.[4]

Definition 2.1[3]

Let *G = (V, E)* be a (simple) graph without isolated vertex. Remember that A_x is the set of all vertices adjacent to x. It is clear that $x \in A_y$ if and only if $y \in A_x$ for all $x, y \in V$ and $x \notin A_x$ for all $x \in V$. Define S_G as follows: $S_G = \{A_x \mid x \in V\}$, Since G has no isolated vertex we have $V = \cup_{x \in V} A_x$. Hence S_G forms a sub basis for a topology τ_G on V called graphic topology of G.

Note[3]: Topologies of K_n and C_n are discrete, but the graphic topology of P_n is not discrete because the set contains two vertices of degree one is not open. Also graphic topology of $K_{n,m}$ is equal to $\{Q, V, A, B\}$ where A and B are partite sets of $K_{n,m}$.

Definition 2.2[4]

Let $G = (V, E)$ be a (simple) graph without isolated vertex. Let I_e be the incidence vertices with the edge e. Define S_{IG} as follows: $S_{IG} = \{I_e \mid e \in E\}$. Since G has no isolated vertex we have $V = \cup_{e \in E} I_e$. So S_{IG} forms a subbasis for a topology τ_{IG} on V called incidence topology of G.

Note[4]: It is obvious that the incidence topologies of Cycle C_n, $n \geq 3$, the Complete graph K_n, $n \geq 3$ and the Complete bipartite graph $K_{n,m}$, $n, m > 1$ are discrete, but the incidence topology of the Path P_n is not discrete because P_n contains two vertices incident with one edge is not open.

Proposition 2.3[3]

Suppose that $G = (V, E)$ is a graph. Then (V, τ_{IG}) is an Alexandroff space.

Proposition 2.4[4]

Suppose that τ_{IG} is the incidence topology of the graph $G = (V, E)$. If $d(v) \geq 2$, then $\{v\} \in \tau_{IG}$ for every $v \in V$.

Proposition 2.5[4]

Let G = (V, E) be a graph. If $d(v) \geq 2$ for all $v \in V$, then τ_{IG} is a discrete topology.

Proposition 2.6[4]

In any graph $G = (V, E)$, $U_V = \cap_{e \in I_v} I_e$ for every $v \in V$.

Remark 2.7[4]

Let $G = (V, E)$ be a graph, then I_v is the set of all edges incident with the vertex v.

Remark 2.8[4]

The Alexandroff topological space (X, τ) is T_1 if and only if $U_x = \{x\}$. It follows that (X, τ) is discrete. Therefore, the incidence topology (V, τ_{IG}) which is Alexandroff space T_1 if and only if it is discrete. If (V, τ_{IG}) is an Alexandroff space, then (V, τ_{IG}) is T_0 space if and only if U_u=U_v implies $u = v$. This means $U_u \neq U_v$ for all distinct pair of vertices $u,v \in V$. The incidence topology is T_0 if and only if $I_u \neq I_v$ for every distinct pair of vertices $u, v \in V$.

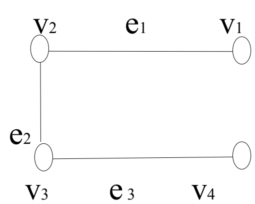
3 AN ALEXANDROFF BITOPOLOGICAL SPACE :

The graphic topology and the incidence topology on V forms the bitopological space (V, τ_G, τ_{IG}). This bitopological Space (V, τ_G, τ_{IG}) is called Alexandroff bitopological space if and only if arbitrary intersection of members of S_G and S_{IG} are open in τ_G and τ_{IG} respectively on V.

Example 3.1

Let $G = (V, E)$ be a simple graph as in figure such that

$V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3\}$



Then S_G=$\{\{v_2\}, \{v_1,v_3\}, \{v_2, v_4\}, \{v_3\}\}$ and S_{IG}=$\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$

$\tau_G = \{\emptyset, V, \{v_2\}, \{v_1,v_3\}, \{v_2, v_4\}, \{v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$ and $\tau_{IG} = \{\emptyset, V, \{v_2\}, \{v_3\}, \{v_1,v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$ Therefore τ_G and τ_{IG} on V give the Alexandroff bitopological space (V, τ_G, τ_{IG}).

Theorem 3.2

Let $G = (V, E)$ be a locally finite graph. Then the bitopological space (V, τ_G, τ_{IG}) is Alexandroff bitopological space.

Proof:

First we have to prove that the arbitrary intersection of members of $S_G$is open in τ_G. If $x \in \cap_{y \in V_1} A_y$, for some subset V_1 of V. This leads that $x \in A_y$ for each $y \in V_1$. Therefore y $\in A_x$ for each y$\in V_1$ and hence $V_1 \subseteq A_x$. Since G is locally finite. A_x and so $V_1$are finite sets, this means that$V_1$is finite, then $\cap_{y \in V_1} A_y$ is empty. But if V_1 is finite, then$\cap_{y \in V_1} A_y$ is the intersection of finitely many open sets and is open in τ_G.

Now to prove that the arbitrary intersection of members of S_{IG} is open in τ_{IG}. If $v \in \cap_{e \in E_1} I_e$

. for some subset E_1 of E. This leads that $v \in I_e$ for each $e \in E_1$. Therefore $e \in I_v$ for each $e \in E_1$ and hence $E_1 \subseteq I_v$ and so v $\in \cap_{e\in I_v} I_e$. Since G is locally finite, therefore I_e and E_1 are finite sets. Then $\cap_{e\in I_v} I_e$ is the intersection of finitely many open sets and is open in τ_{IG}.

Theorem 3.3

The Alexandroff bitopological space (V, τ_G, τ_{IG}) of a graph $G = (V, E)$ is pair wise T_0.

Proof:

Let *(x, y)* be the any distinct pair of vertices of V. Now there exists two cases

Case:1 If the nodes x and y are adjacent, then by the definition of graphic topology τ_G there are two τ_G-open sets U_x and U_y such that U_x containing x but not y and U_y containing y but not x, where U_x is the intersection of all open set containing x is that the smallest open set and U_y is the intersection of all open sets containing y is that the smallest open set.

Case:2 If the nodes x and y are not adjacent, then there exists two different edges e_1, $e_2 \in E$ such that x incident with e_1 and y incident with e_2. Then by the definition of incidence topology τ_G, there are two τ_{IG} open sets U_x and U_y such that U_x containing x but not y and U_y containing y but not x. From above cases, for each pair of distinct nodes (x, y) of V, there is either a τ_G open set containing x but not y or there exists a τ_{IG} open set containing y but not x. Hence the Alexandroff bitopological space *(V,* τ_G, τ_{IG}*)* is pair wise T_0.

Theorem 3.4

Let $G = (V, E)$ be a locally finite graph. Then the Alexandroff bitopological space *(V,* τ_G, τ_{IG}*)* is weakly pair wise T_0.

Proof:

Let $G = (V, E)$ be a simple graph. If (V, τ_{IG}) is Alexandroff space then *(V,* τ_{IG}*)* is T_0 space if and only if $U_x \neq U_y$, for every distinct pair of nodes $x, y \in V$, where U_x is the intersection of all open set containing x is that the smallest open set and U_y is the intersection of all open sets containing y is that the smallest open set. Suppose $U_x = U_y$, for every distinct pair of nodes $x, y \in V$ in τ_{IG}, then which implies that $I_x = I_y$ for all distinct pair of nodes $x, y \in V$. This leads that x and y are adjacent nodes of degree one. Then by the definition of graphic topology $\{x\}, \{y\} \in \tau_G$. Therefore, for every pair of distinct nodes of V, there exists a τ_G open set or τ_{IG} open set containing one but not the other. Hence the Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise

T_0.

Theorem 3.5

The Alexandroff bitopological space *(V, τ_G, τ_{IG})* of a graph $G = (V, E)$ is weakly pair wise T_1. if and only if $U_x \neq U_y$ in τ_G and τ_{IG}, for every distinct pair of nodes $x, y \in V$.

Proof:

Let us assume Alexandroff bitopological space *(V, τ_G, τ_{IG})* be a weakly pairwise T_1. By contradiction, suppose that there exists a distinct pair of nodes $x, y \in V$ such that $U_x = U_y$ in τ_G or τ_{IG} .

Case:1 If $U_x = U_y$ in τ_G , which implies that $A_x = A_y$ then τ_G is not a T_0 space. That is there is no τ_G open set containing *x* but not *y* or containing *y* but not *x*. Which is contradiction to Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_1. .

Case:2 If $U_x = U_y$ in τ_{IG} , which implies that $I_x = I_y$ then τ_{IG} is not a T_0space. That is there is noτ_{IG} open set containing *x* but not *y* or containing *y* but not *x*. Which is contradiction to Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_1.

Conversely, let us assume that $U_x \neq U_y$ in τ_G and τ_{IG} for every distinct pair of nodes $x, y \in V$. Now we have two cases,

Case:1 If the nodes *x* and *y* are adjacent then by the definition of graphic topology there are two τ_G open sets U_x and U_y such that U_x containing *x* but not *y* and $U_y$containing *y* but not *x*. From our assumption $U_x \neq U_y$ in τ_G for every distinct pair of nodes $x, y \in V$ in τ_{IG} , so that τ_{IG} is T_0space. That is there exists τ_{IG} open set containing *x* but not *y* or containing *y* but not *x*.

Case:2 The nodes *x* and *y* are not adjacent. This means that there exists two different edges $e_1, e_2 \in E$ such that *x* incident with e_1 and *y* incident withe_2. Then by the definition of incidence topology τ_{IG} there are two τ_{IG} open sets such that U_x containing *x* but not *y* and U_y containing *y* but not *x*. From our assumption $U_x \neq U_y$ for every distinct pair of nodes $x, y \in V$ in τ_G is T_0space. That is there exists aτ_G open set containing *x* but not *y* or containing *y* but not *x*. From cases above, for each pair of distinct nodes $x, y \in V$, there exists a τ_{IG} open set *D* and τ_{IG} open set *W* such either that $x \in D, y \notin D$ and $x \notin W, y \in W$ or $x \in W, y \notin W$ and $x \notin D, y \in D$. Hence the Alexandroff bitopological space *(V, τ_G, τ_{IG})* of a graph $G = (V, E)$ is weakly pair wise T_1.

Theorem 3.6

The Alexandroff bitopological space (V, τ_G, τ_{IG}) of a graph $G = (V, E)$ is weakly pair wise T_1 if and only if τ_G and τ_{IG} are discrete topologies.

PROOF:

The Alexandroff bitopological space (V, τ_G, τ_{IG}) is pair wise T_1 if and only if τ_G and τ_{IG} is T_1, because pair wise T_1 in each topology if and only if τ_G and τ_{IG} are discrete topologies[8]. Since from [remark 2.8 τ_G and $\tau_{IG}$are T_1 spaces if and only if τ_G and $\tau_{IG}$are discrete topologies.

Theorem 3.7

Let $G = (V, E)$ be a locally finite graph. Then the Alexandroff bitopological space $(V, \tau_G, \tau_{IG}$ $)$ is pair wise T_2 if and only if for every distinct pair of nodes $x, y \in V$ such that $U_x \neq U_y$ in τ_G and τ_{IG} and the length of any Path is $P \geq 4$.

Proof:

Let us assume that the Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_2. By contradiction, suppose that there exists a distinct pair of nodes $x, y \in V$ such that $U_x = U_y$ in τ_G or τ_{IG} or there is a path of length less than four between two distinct pendent nodes.

Case:1 $U_x = U_y$ in τ_G which implies that $A_x = A_y$ in τ_G. Therefore τ_G is not a T_0 space. That is there is no τ_G open set containing x but not y or containing y but not x. Which is contradiction to Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_2.

Case:2 $U_x = U_y$ in τ_{IG} which implies that $I_x = I_y$, Then τ_{IG} is not a T_0 space. That is there is no τ_{IG} open set containing x but not y or containing y but not x. Which is contradiction to Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_2.

Case:3 Suppose that x and y are two distinct pendent nodes and P is the path of length less than four between x and y. If the Path of length is one or two then $U_x = U_y$ in τ_G or τ_{IG} respectively. which is contradiction with the assumption. Suppose the Path of length of the length is three, let the Path $P = xe_1ue_2ve_3y$, where $x, u, v, y \in V$ and $e_1, e_2, e_3 \in E$. Then the open set in τ_G that contain x and y are U_x and U_y respectively, such that $x, v \in U_x$ and $u, y \in U_y$ And also the open set in τ_{IG} that contain x and y are U_x and U_y respectively, such that $x, u \in U_x$ and $v, y \in U_y$. Thus we have $A_u \cap I_{e_3}$ $= \varnothing$ and $A_v \cap I_{e_3} = \varnothing$. So there is no τ_G open set D and τ_{IG} open set W such that $x \in D$ and $y \in W$ or $x \in W$ and $y \in D$, which is contradiction with the assumption. Since the Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_2.

conversely, let us assume that $U_x \neq U_y$ for every distinct pair of nodes $x, y \in V$ in τ_G and τ_{IG} and the length of any path between any two distinct pendent nodes is at least four. Let us consider, any pair of distinct nodes $x, y \in V$, we have three cases.

Case:1 x and y are adjacent nodes such that $d(x) = 1$ and $d(y) \geq 2$ then $y \in \tau_{IG}$. From the definition of graphic topology U_y is an open set containing y but not x. This implies that A_y is open set containing x and but not y. Hence A_y is τ_G open set containing x and $\{y\}$ is τ_{IG} open set containing y such that $A_y \cap \{y\} = \varnothing$.

Case:2 x and y are not adjacent nodes of degree at least two. We have $\{x\}, \{y\} \in \tau_{IG}$. From our assumption $U_x \neq U_y$ for every distinct pair of nodes $x, y \in V$ in τ_G. This implies that $A_x \neq A_y$ for every distinct pair of nodes $x, y \in V$. Then τ_G is T_0 space. That is there exists a τ_G open set containing x but not y or containing y but not x. Therefore, there exists a τ_G open set D and τ_{IG} open set W with $D \cap W = \varnothing$ such that $x \in D, y \in W$ or $x \in W$ and $y \in D$.

Case:3 x and y are not adjacent nodes, such that $d(x) = 1$ and $d(y) \geq 2$. Suppose that $u \in V$ is a node adjacent with x. This means there exists an edge $e \in E$ such that $e = xu$. Now either y adjacent with u or y is not adjacent with u.

a) If y is adjacent with u then $I_e = \{x, u\}$ isτ_G open set by the definition of incidence topology. Since y is adjacent with u from graphic topology τ_G we have $x, u \in U_y$. Hence I_e is τ_{IG} open set containing x and U_y is τ_G open set containing y such that $I_e \cap U_y = \varnothing$.

b) If y is not adjacent with u then U_u is an open set containing u but not y in τ_G. Which implies A_u is an open set containing x but not y by the definition of graphic topology τ_G. So we have $\{y\} \in \tau_{IG}$ .Therefore, $A_u \in \tau_G$ open set containing x and $\{y\}$ is τ_{IG} open set containing y such that $A_u \cap \{y\} = \varnothing$.

Case:4 x and y are not adjacent nodes such that $d(x) = 1$ and d(y) = 1. From assumption the length of any path between x and y is at least four. Let $P = xe_1ue_2ve_3ye_4z$ be the path between x and y such that $x, u, v, y, z \in V$ and $e_1, e_2, e_3, e_4 \in E$ then by the definition of graphic topology and incidence topology A_u and I_{e_4} are τ_G open set and τ_{IG} open set respectively, $A_u \cap I_{e_4} = \varnothing$ such that $x \in A_u$ and $y \in I_{e_4}$Form cases above, for each pair of nodes $x, y \in V$ there exists a τ_G open set D and τ_{IG} open set W with $D \cap W = \varnothing$ such that $x \in D$, $y \in W$ or $x \in W, y \in D$. Hence the Alexandroff bitopological space (V, τ_G, τ_{IG}) is weakly pair wise T_2.

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