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Fixed Point Theorem for Quadruple Self-Mappings in Digital Metric Space

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ABSTRACT:

In this paper, we have established a common fixed point theorem for quadruple self mappings satisfying the continuous function in digital metric space. Our result is the extension of the results obtained by Jain for complete digital metric spaces. We have concluded an example to support our main result.

KEYWORDS: Common Fixed point, Digital metric space, Quadruple mappings, continuous functions.

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INTRODUCTION:

It was the dawn of the fixed point theory when in 1912 Brouwer proved a fixed point result for continuous self maps on a closed ball then later in 1922, Banach[1] gave a very useful result in form of Banach Contraction Principle. At first it was Rosenfeld[2] who considered digital topology as a tool to study digital images. The digital version of the topological concepts was produced by Boxer[3] and later he studied digital continuous functions[4]. Ege and Karaca[5] established relative and reduced Lefschetz fixed point theorem for digital images, the notion of a digital metric space was also proposed by them. Further, the famous Banach Contraction Principle for digital images was proved by Ege[6]. In digital topology the notion of digital continuity was developed by Rosenfeld[7] to study 2D and 3D digital images. Further in 2013, Ege and Karaca[5] described the digital continuous functions.

Fixed point theory has applications in field of mathematics, computer science, engineering, game theory, fuzzy theory, image processing and so forth. In metric space it begins with the Banach fixed-point theorem providing a constructive method to find fixed points and is an essential tool to find solution of some problems in mathematics and engineering and consequently has been generalized in many ways. Till now, several developments have occurred in this area. In 1976 a major shift in the arena of fixed point theory came when Jungck[8,9,10], defined the concept of commutative and compatible maps and proved the common fixed point results for such maps. Later on, Sumitra et.al[11] proved common fixed point theorems for compatible map in digital metric space with its applications. Certain alterations of commutativity and compatibility can also be found in refrences[9,10,12]. The notion of compatible mappings of type(R) in digital metric space was introduced by Jain[13]. In this paper we have generalized and extended the results obtained by Jain[13] for quadruple mappings in complete digital metric space.

2. DEFINITIONS:

Definition 2.1.[4] Let $(X,\rho_0) \subset \mathbb{Z}^{n_0}, (Y,\rho_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f: X \to Y$ be a function.

(i) If for every ρ_0 −connected subset U of X, $f(U)$ *is a* ρ_1 −connected subset of Y, *then* f is said to be (ρ_0,ρ_1) −continuous.

(ii) f *is* (ρ_0,ρ_1) −continuous for every ρ_0 −adjacent points $\{x_0,x_1\}$ of X, either $f(x_0) = f(x_1)$ *or* $f(x_0)$ *and* $f(x_1)$ *are* ρ_1 − adjacent in Y.

(iii) If f *is* (ρ_0,ρ_1) −continuous, bijective and f^{-1} is (ρ_0,ρ_1) −continuous, then f is called (ρ_0,ρ_1) −isomorphism and denoted by $\cong (\rho_0,\rho_1)\, Y$.

Proposition 2.2.[6] Every digital contraction map $T: (X,d,\rho) \to (X,d,\rho)$ is digitally continuous.

Definition 2.3. [5] Let $X \subseteq Z^n$ and (X, d, ρ) be a digital metric space. Then there does not exist a sequence $\{x_n\}$ of distinct elements in X, such that

$$d(x_{m+1}, x_m) < d(x_m, x_{m-1}) \quad for\ m = 1,2,3, \dots$$

Definition 2.4. [4] Suppose that (X, d, ρ) is a digital metric space and $P, Q: X \to X$, and be two self-maps defined on X. then $P\ and\ Q$ are compatible if

$$d(PQx, Qpx) \leq d(Px, Qx)\ for\ all\ \ x \in X.$$

Definition 2.5. [14] Let (P, Q) be a pair of self-mappings on a metric space (X, d) and $x \in X$. then P is called (G, O)- continuous at X if $PX_n \to PX$, for every sequence $\{x_n\} \subset X$ with $Qx_n \uparrow Qx$ $(resp.\ Qx_n \downarrow Qx,\ Qx_n \uparrow\downarrow Qx)$. Moreover, P is called (G, O)- continuous (resp.($(G, \bar{O}) - continuous$, (G, O)- continuous) at every point of $x \in X$.

Remark 2.6. [14] In digital metric space, G-continuity⇒ $(G, O) - continuity \Rightarrow (G, \bar{O}) - continuity$ (as well as $(G, \underline{O}) - Continuity$).

Proposition 2.7. [6] Let (X, d, ρ) is a digital metric space. A sequence $\{x_0\}$ of points of a digital metric space (X, d, ρ) is

(i) A Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that for all, $n, m \geqq \alpha$, then

$$d(x_n, x_m) \leqq 1$$

i.e., $x_n = x_m$.

(ii) Convergent to a point $l \in X$ if for all $\epsilon \geqq 0$, there is $\alpha \in \mathbb{N}$ such that for all $n \geqq \alpha$ then $d(x_n, l) \leqq \epsilon, i.e, x_n = l$.

Lemma 2.8. [6] Let $\{x_n\}$ be a sequence of complete digital metric space (X, d, ρ). If there exists $\alpha \in (0,1)$ such that $d(x_{n+1}, x_n) \leq \alpha d(y_n, y_{n-1})\ \forall\ n, then\ \{x_n\}\ converges\ to\ a\ point\ in\ X.$

Proposition 2.9. [6] A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $n \geqq \alpha, then\ x_n = l.$

Proposition 2.10. [13] Let f and g be digitally compatible mappings of type (R) of a digital metric space (X, d, ρ) into itself. Suppose that $\lim_n f\,x_n = \lim_n g\,x_n = t\ for\ some\ t \in X. then$

(a) $\lim_n gf\,x_n = ft\ if\ f$ is digitally continuous at t.

(b) $\lim_n fg\,x_n = gt\ if\ g$ is digitally continuous at t.

(c) $fgt = gft\ \ and\ ft = gt\ if\ f\ and\ g\ both\ are\ digitally\ continuous\ at\ t.$

3. MAIN RESULTS.

Theorem 3.1: Let $P, Q,\ G$ and H be quadruple mappings of a complete digital metric space (X, d, ρ) satisfying the conditions.

1) $G(X) \subset Q(X), H(X) \subset P(X),$

2) $d(x,y) = \alpha \max\left\{d(Gx,Hy), d(Gx,Px), d(Hy,Qy), d(Hx,Gy), d(Hx,Qx), \frac{1}{2}\big(d(Gx,Qy) + d(Hy,Px)\big)\right\}$ for all $x,y \in X$, where $\alpha \in (0,1)$.

3) In mappings P, Q, G and H one of the mappings is continuous.

Assume the pairs (P,G) and (Q,H) are compatible, then P, Q, G and H have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $Q(X) \subset G(X)$ and $P(X) \subset H(X)$, we can construct the sequence $\{y_n\}$ in X such that,

$y_{2n} = G(x_{2n}) = Q(x_{2n+1})$ and $y_{2n+1} = H(x_{2n+1}) = P(x_{2n+2})$ for each $n \geq 0$.

$d(y_{2n}, y_{2n+1}) = d\big(G(x_{2n}), H(x_{2n+1})\big)$

$$\leq \alpha \max \left\{ \begin{array}{c} d\big(P(x_{2n}), Q(x_{2n+1})\big), d\big(P(x_{2n}), G(x_{2n})\big), d\big(Q(x_{2n+1}), H(x_{2n+1})\big), d\big(G(x_{2n}), H(x_{2n+1})\big), \\ \frac{1}{2}\Big(d\big(G(x_{2n}), Q(x_{2n+1})\big) + d\big(P(x_{2n}), H(x_{2n+1})\big)\Big) \end{array} \right\}$$

$\leq \alpha \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}) \frac{1}{2}\big(d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\big)\right\}$.

$\leq \alpha \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n+1}), d(y_{2n+1}, y_{2n}), \frac{1}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))\right\}$

Putting $d_n = d(y_n, y_{n+1})$, we have

$$d_{2n} \leq \alpha \max\left\{d_{2n-1}, d_{2n}, \frac{1}{2}(d_{2n-1} + d_{2n})\right\}.$$

Now let $d_{2n} > d_{2n-1}$.

Therefore, $d_{2n} \leq 2\alpha\, d_{2n}$, for all $\alpha \in (0,1)$, which is a contradiction.

Hence, $d_{2n} \leq d_{2n-1}$.

Let $m, n \in N$ such that $m > n$, we have

$$d(y_m, y_n) \leq \alpha d(y_m, y_{m-1}) + \cdots + \alpha d(y_{n+1}, y_n)$$

$\leq \alpha^n\big(d(y_1, y_0)\big) \to 0$ as $m, n \to \infty$.

It implies $\{y_n\}$ is digitally Cauchy sequence in digital metric space (X, d, ρ) and converges to point z as $n \to \infty$. Similarily sub sequences $\{Gx_{2n}\}, \{Px_{2n}\}, \{Hx_{2n+1}\}$ and $\{Qx_{2n+1}\}$ also converges to the point z.

Now suppose P is continuous. Since P and G are digitally compatible mapping then from Proposition 2.10 the sequence $\{PPx_n\}$ and $\{GPx_{2n}\}$ converges to Pz as $n \to \infty$.

Now we claim that $z = Pz$, for this on putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition 2 we have

$$d(GPx_{2n}, Hx_{2n+1}) \leq \alpha \max \left\{ \begin{matrix} d(PPx_{2n}, Qx_{2n+1}), d(PPx_{2n}, GPx_{2n}), d(Qx_{2n+1}, Hx_{2n+1}), d(PPx_{2n}, Gx_{2n+1}), \\ \frac{1}{2}\left(d(GPx_{2n}, Qx_{2n+1}) + d(PPx_{2n}, Hx_{2n+1})\right) \end{matrix} \right\}.$$

Letting $n \to \infty$, we have

$$d(Pz, z) \leq \alpha \max\left\{d(Pz, z), d(Pz, Pz), d(z, Pz), d(Pz, Pz), d(z, z), \frac{1}{2}\left(d(Pz, z) + d(Pz, z)\right)\right\}$$

$$\leq \alpha d(Pz, z)$$

It implies that $Pz = z$.

Now we claim that $Gz = z$. Putting $x = z$, $y = x_{2n+1}$ in condition2 we have

$$d(Gz, Hx_{2n+1}) \leq \alpha \max \left\{ \begin{matrix} d(Pz, Qx_{2n+1}), d(Px, Gx_{2n+1}), d(Qx_{2n+1}, Hx_{2n+1}), d(Px, Hx_{2n+1}), d(Gx_{2n+1}, Qx_{2n+1}) \\ \frac{1}{2}\left(d(Gz, Qx_{2n+1}) + d(Pz, Hx_{2n+1})\right) \end{matrix} \right\}.$$

Letting $n \to \infty$, we have

$$d(Qz, z) \leq \alpha \max\left\{d(z, z), d(z, Gz), d(z, Hz), d(z, Qz), d(z, z)\frac{1}{2}\left(d(Gz, z) + d(Gz, z)\right)\right\} \leq \alpha d(Gz, z)$$

It implies that $Pz = z$.

Since $G(X) \subset Q(X)$ and hence there exists a point u in X such that $z = Gz = Qu$.

Now we claim that $z = Hu$.

$$d(z, Hu) = d(Gz, Hu)$$

$$\leq \alpha \max\left\{d(Pz, Qu), d(Pz, Gz), d(Qu, Hu), d(Pz, Hu), d(Pu, Qu)\frac{1}{2}\left(d(Gz, Qu) + d(Pz, Hu)\right)\right\}$$

$$\leq \alpha \max\left\{d(z, z), d(z, z), d(z, Hu), d(z, Qu), d(z, Gu)\frac{1}{2}\left(d(z, z) + d(z, Hu)\right)\right\}.$$

It implies that $z = Hu$.

Since (Q, H) is compatible and $Qu = Hu = z$, by proposition 2.7 we have $d(QHu, HQu) = 0$ and hence $Qz = QHu = HQu = Hz$. Also from condition 2, we have

$$d(z, Qz) = d(Gz, Hz)$$

$$\leq \alpha \max\left\{d(Pz, Qz), d(Pz, Gz), d(Pz, Hz), d(Gz, Hz)d(Qz, Hz), \frac{1}{2}\left(d(Gz, Qz) + d(Pz, Hz)\right)\right\}$$

$$\leq \alpha \max\left\{d(z, Qz), d(z, z), d(z, Hz), d(z, Gz), d(Qz, Qz), \frac{1}{2}\left(d(z, Qz) + d(z, Qz)\right)\right\}.$$

It implies that $z = Qz$.

Hence $z = Qz = Hz = Pz = Gz$.

Therefore P, Q, G and H have common fixed point z.

Similarly, proof can also be completed taking Q as continuous.

Now suppose G is continuous. Since P and G are compatible by Proposition 2.10 we have GGx_{2n} and PGx_{2n} converges to Gz as $\to \infty$.

We claim that $z = Gz$ and on putting $x = Gx_{2n}$, $y = x_{2n+1}$ in condition 2 we have,

$$d(GGx_{2n}, Hx_{2n+1}) \leq \alpha \max \begin{Bmatrix} d(PGx_{2n}, Qx_{2n+1}), d(PGx_{2n}, GGx_{2n}), d(PGx_{2n}, Hx_{2n+1}), d(PGx_{2n}, HHx_{2n}), d(Qx_{2n+1}, Hx_{2n+1}), \\ \frac{1}{2}\left(d(GGx_{2n}, Qx_{2n+1}) + d(PGx_{2n}, Hx_{2n+1})\right) \end{Bmatrix}$$

Letting $n \to \infty$, we have

$$d(Gz, z) \leq \alpha \max \left\{ d(Gz, z), d(Gz, Gz), d(Gz, Hz), d(Hz, Gz), d(z, z), \frac{1}{2}\left(d(Gz, z) + d(Gz, z)\right) \right\}$$
$$= \alpha d(Gz, z),$$

It implies $Gz = z$.

Since $GX \subset QX$, hence there exists a point w in X such that $z = Gz = Qw$.

We claim that $z = Hw$ and on putting $x = Gx_{2n}, y = w$ in condition 2 we have

$$d(GGx_{2n}, Hw) \leq \alpha \max \begin{Bmatrix} d(PGx_{2n}, Qw), d(PGx_{2n}, GGx_{2n}), d(PGx_{2n}, HHx_{2n}), d(PGx_{2n}, HGx_{2n}), d(Qw, Hw), \\ \frac{1}{2}\left(d(GGx_{2n}, Qw) + d(PGx_{2n}, Hw)\right) \end{Bmatrix}$$

i.e.

$$d(z, Hw) \leq \alpha \max \left\{ d(z, z), d(z, z), d(z, z), d(z, z), d(z, Hw), \frac{1}{2}\left(d(z, z) + d(z, Hw)\right) \right\}$$

it implies that $z = Hw$.

Since Q and H are compatible on X and $Qw = Hw = z$, so by Proposition 2.7 $d(QHw, HQw) = 0$.

Hence $Qz = QHw = HQw = Hz$.

Now we claim that $z = Hz$, on putting $x = x_{2n}$, $y = z$ in condition 2 we have

$$d(Gx_{2n}, Hz) \leq \alpha \max \begin{Bmatrix} d(Px_{2n}, Qz), d(Px_{2n}, Gx_{2n}), d(Px_{2n}, Hx_{2n}), d(Px_{2n}, Hz) d(Qz, Hz), \\ \frac{1}{2}\left(d(Gx_{2n}, Qz) + d(Px_{2n}, Hz)\right) \end{Bmatrix}$$

i.e.

$$d(z, Hz) \leq \alpha \max \left\{ d(z, Hz), d(z, z), d(z, z), d(z, z), d(Hz, Hz), \frac{1}{2}\left(d(z, Hz) + d(z, Hz)\right) \right\}$$

it implies that $Hz = z$.

Since $HX \subset PX$, there exists a point p in X such that $z = Hz = Pp$.

We claim that $z = Gp$, on putting $x = p, y = z$ in condition 2 we have

$$(Gp, z) = d(Gp, Hz) \leq \alpha \max \left\{ \begin{matrix} d(Pp, Qz), d(Pp, Gp), d(Pp, Qz), d(Pp, Hz), d(Qz, Hz), \\ \frac{1}{2}\left(d(Gp, Qz) + d(Pp, Hz)\right) \end{matrix} \right\}$$

$$\leq \alpha \max \left\{ d(z, z), d(z, Gp), d(z, Gz), d(z, z), d(Hz, Hz), \frac{1}{2}\left(d(Gp, z) + d(z, z)\right) \right\},$$

It implies that $Gp = z$.

Since, P and G are compatible on X, $Gp = Pp = z$. So, by proposition 2.7, we have $d(PGp, GPp) = 0$ and hence $Pz = PGp = GPp = Gz$.

That is, $z = Pz = Gz = Qz = Hz$. Therefore, z is common fixed point of P, G, Q and H.

Similarily, proof can be completed when H is continuous.

Hence, uniqueness follows easily.

Example 3.2. Let $X = [0, \infty)$ be a digital metric space, and $M(x, y, \rho) = \left(\frac{\rho}{\rho+1}\right) d(x, y)$, *where* $d(x, y) = |x - y|$. Let $P: X \to X$ be a self-mapping defined by

$$P(x) = \begin{cases} \frac{x}{6} & if \ \ x \leq 1 \\ 0 & if \ x \geq 1 \end{cases}$$

We can easily that P is a continuous mapping.

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