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Even distance closed domination critical graphs

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ABSTRACT

In a graph G = (V, E), a subset $S \subset V(G)$ is said to be a distance closed dominating (D.C.D) set, if (i) <S> is distance closed; (ii) S is a dominating set. A distance closed dominating set D is said to be an Even Distance Closed Dominating (E.D.C.D) set, if for any vertex $u \in V\text{-}D$, there exists a vertex $v \in D$ at even distance from u. The cardinality of a minimum even distance closed dominating set is called an even distance closed domination number and it is denoted by $\gamma_{edcl.}$ In this paper, the even distance closed domination critical graph with respect to edge addition is defined and through which the structural properties of graphs are studied.

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KEYWORDS: distance closed dominating set, even distance closed dominating set, even distance closed domination critical graphs.

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1. INTRODUCTION

The concept of domination in graphs was introduced by Ore[1] in 1962. It is originated from the chess game theory, which paved the way to the development of the study of various domination parameters and then relation to various other graph parameters. A set $D \subseteq V(G)$ is called a dominating set of G if every vertex in V(G) – D is adjacent to some vertex in D and D is said to be a minimal dominating set if D – {v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set, if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. The concept of dominating set and different types of dominating set are studied [2]. At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad-hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest.

Graphs, which are critical with respect to a given property frequently, play an important role in the investigation of that property. A graph G is said to be domination critical if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$, we will say G is k-domination critical or just k-critical. The concept of domination critical graphs and their structural properties are studied [3, 4]. The critical concept in graphs plays an important role in the study of structural properties of graphs and hence it will be useful to study any communication model. In this paper, we introduced a new domination critical graph called even distance closed domination critical graphs through which the structural properties of those graphs are studied.

2. PRIOR RESULTS

The concept of ideal set is defined and studied by Janakiraman [5]. The ideal set without minimality condition is taken as a distance closed set and the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a *distance closed dominating (D.C.D) set,*

(i) <S> is distance closed, for each vertex u ε S and for each w ε V – S, there exists at least one vertex v ε S such that $d_{<S>}(u, v) = d_G(u, w)$ and

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the *distance closed domination number* of G and is denoted by γ_{dcl}. The definition and the extensive study of the above said distance closed domination is studied [6]. A graph G is said to be a *distance closed domination critical* if for every edge $e \notin E(G)$, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$. If G is a D.C.D critical graph with $\gamma_{dcl}(G) = k$, then G is said to be k-D.C.D critical. Also the structural properties of any k-D.C.D critical graph G with type I(structure having every minimum D.C.D set is a path of length k) and type II(structure having every minimum D.C.D set is a cycle of length k) are analyzed [7, 8]. The following results given [7, 8] are used to prove many results in the present work.

Theorem 2.1: G is 3-D.C.D critical if and only if

(i) G is connected.

(ii) G has $\gamma_{dcl}(G) = 3$.

(iii) G has exactly one vertex with eccentricity equal to 1.

(iv) For every pair of non-adjacent vertices at least one of them is of degree $p - 2$.

Theorem 2.2: A graph G is 4-D.C.D critical if and only if

(i) G is connected.

(ii) G has $\gamma_{dcl}(G) = 4$.

(iii) For any two non-adjacent vertices at least one of them is of degree $p - 2$.

Theorem 2.3: If G is any k-D.C.D type (I) critical graph then

i. G is of diameter $k - 1$ and radius $k - \frac{1}{2}$.

ii. G has at most two pendant vertices.

iii. G can have at most $(k - 2)$ cut vertices.

iv. G is diameter edge (addition) critical.

v. G is also a block and is Hamiltonian.

Theorem 2.4: If G is any k-D.C.D type (II) critical graph then

(i) G is $\frac{k}{2}$ self-centered.

(ii) G is a block.

(iii) G is radius edge (addition) critical.

(iv) G is Hamiltonian.

3. MAIN RESULTS

A distance closed dominating set D is said to be an ***E**ven **D**istance **C**losed **D**ominating (E.D.C.D) set*, if for any vertex $u \in V\text{-}D$, there exists a vertex $v \in D$ at even distance from u. The cardinality of a minimum even distance closed dominating set is called an *even distance closed domination number* and it is denoted by γ_{edcl}. For example, in the graph G given in Figure 3.1, the set $S = \{1, 3, 5, 6\}$ forms an E.D.C.D set and $\gamma_{edcl}(G) = 4$.

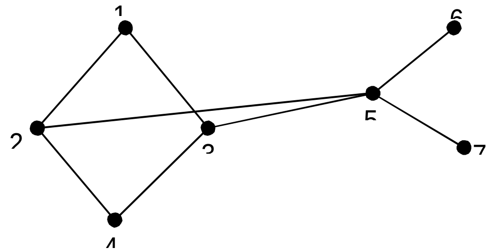

Figure 3.1 - An example of E.D.C.D set of a graph

For any graph G, $\gamma_{edcl} \geq \gamma_{dcl}$. There are some graphs in which $\gamma_{edcl} = p$ and are called 0-even distance closed dominating graphs. For example, complete graphs are 0-even distance closed dominating graphs. The definition and the extensive study of the even distance closed domination are studied [9].

Continuing the above, we studied the critical concept of the even distance closed domination in graphs while adding an edge in that graph. The even distance closed domination critical graph is defined as follows:

A graph G is said to be an *even distance closed domination (E.D.C.D) critical* if for every edge $e \notin E(G)$, $\gamma_{edcl}(G + e) < \gamma_{edcl}(G)$. If G is an E.D.C.D critical graph with $\gamma_{edcl}(G) = k$, then G is said to be k-E.D.C. D critical.

3.1 Structural properties of k-E.D.C.D, k ≤ 6 critical graphs

Proposition 3.1.1: There is no 2-E.D.C.D critical graph.

Proposition 3.1.2: There cannot be any 3-E.D.C.D critical graph.

Proof: Suppose that, if G is a 3-E.D.C.D critical graph, then $\gamma_{edcl}(G) = 3$ and G must have exactly one vertex with eccentricity equal to 1 and $\gamma_{edcl}(G + e) = 2$, for any edge $e \notin E(G)$. That is, the E.D.C.D set D of (G + e) has two vertices and every vertex in V – D is at an even distance from at least one vertex of D, which is a contradiction, as the vertices in D are of eccentricity equal to 1 and every vertex in V – D is adjacent to both the vertices of D. Hence, there cannot be any 3-E.D.C.D critical graph.

Theorem 3.1.1: There cannot be any 4-E.D.C.D critical graph.

Proof: Let G be a 4-E.D.C.D critical graph and let D be an E.D.C.D set of G. Then clearly $\gamma_{edcl}(G) = 4$ and $\gamma_{edcl}(G + uv) \leq 3$, for every pair of non-adjacent vertices u and v of G. In order to prove G cannot be 4-E.D.C.D critical, we need to prove the following claims:

Claim 1: ***If G is a 4-E.D.C.D critical graph, then G must be 2 self-centered.***

Suppose that, if G is a 4-E.D.C.D critical graph with diameter 3, then addition of an edge between any two peripheral nodes, say u and v will give us a 2 self-centered graph. Thus $\gamma_{edcl}(G + uv) \geq 4$, a contradiction to G is 4-E.D.C.D critical. Hence, G must be 2 self-centered.

Claim 2: ***For every pair of non-adjacent vertices u and v, $\gamma_{edcl}(G + uv) = 3$ and $\gamma_{edcl}(G + uv)$ cannot be equal to 2.***

Suppose that, if $\gamma_{edcl}(G + uv) = 2$ then the E.D.C.D set D^1 of $(G + uv)$ has the two vertices u and v. That is, $D^1 = \{u, v\}$ where $e(u) = e(v) = 1$ in $<G + uv>$. Thus, every vertex in $V – D^1$ is adjacent to all the vertices of D^1 (that is, every vertex is at odd distance from D^1), a contradiction to D^1 is an E.D.C.D set of $(G + uv)$.

Hence, $\gamma_{edcl}(G + uv) = 3$.

Claim 3: ***In G, $\Delta = p – 2$ and $\delta \leq p – 3$.***

By claim 2, $\gamma_{edcl}(G + uv) = 3$. Therefore, $<G + uv>$ must have a vertex of degree $p – 1$. That is, $d(u)$ or $d(v) = p – 1$ in $<G + uv>$. Hence $d(u)$ or $d(v) = p – 2$ in G. Also, both $d(u)$ and $d(v)$ cannot be equal to $p – 1$ in $<G + uv>$ as $\gamma_{edcl}(G + uv) = 3$. Hence, either $d(u)$ or $d(v)$ must be less than or equal to $p – 3$ and hence in G, $\Delta = p – 2$ and $\delta \leq p – 3$.

Claim 4: ***For every pair of non-adjacent vertices u and v in G, exactly one of them is of degree p – 2***

As $<G + uv>$ has exactly one vertex of degree equal to $p – 1$, either u or v has degree $p – 2$ in G not both. Hence, for every pair of non-adjacent vertices u and v in G, exactly one of them is of degree $p – 2$.

Claim 5: ***The set of vertices with degree p – 2 forms a clique in G***

By claim 4, for every pair of non-adjacent vertices u and v in G, exactly one of them is of degree $p – 2$. That is, there is no two non-adjacent vertices u and v in G such that both of them are of degree $p – 2$. Hence, the set of vertices with degree $p – 2$ forms a clique in G.

Claim 6: ***The set of vertices with degree less than or equal to p – 3 forms a clique in G***

By claim 4, for every pair of non-adjacent vertices u and v in G, exactly one of them is of degree $p – 2$. That is, there is no two non-adjacent vertices u and v in G such that both of them are of degree less than or equal to $p – 3$. Hence, the set of vertices with degree less than or equal to $p – 3$ forms a clique in G.

Proof of main theorem:

Let C_1 be the set of vertices with degree $p – 2$ in G and let C_2 be the set of vertices with degree less than or equal to $p – 3$ in G. By claim 5 and 6, $<C_1>$ is a clique and $<C_2>$ is also a clique.

Now, let $u \in C_1$ and $v \in C_2$. Then $d(u) = p – 2$ and $d(v) \leq p – 3$. If we add an edge between u and v then the set $D = \{u, v, x\}$ forms a D.C.D set of $(G + uv)$ where $x \in C_1$ and x is non-adjacent to v. Since $d(x) = p – 2$ and it is non-adjacent to v, every vertex in $C_2 – v$ is adjacent to all the vertices of D. That is, every vertex in $C_2 – v$ is at odd distance from D. As u and v are arbitrary, D cannot be an E.D.C.D set of $(G + uv)$ and hence G cannot be a 4-E.D.C.D critical graph.

Theorem 3.1.2: Any 5-D.C.D type (I) critical graph is 5-E.D.C.D type (I) critical.

Proof: Let G be a 5-D.C.D type (I) critical graph and let D be a minimum D.C.D set of G. Then <D> is the diametral path of G and every vertex in V – D has at least one eccentric vertex in D as D contains the unique pair of peripheral nodes. Hence, every vertex in V – D is non-adjacent to at least one vertex of D and hence D is also an E.D.C.D set of G.

Let u and v be any two non-adjacent vertices of G. Then, clearly (G + uv) is of diameter 3 or it is 2 self-centered. Thus, $\gamma_{dcl}(G + uv) = 4$ and if D^1 is the D.C.D set of (G + uv), then $<D^1>$ is a path of length 4 and every vertex in $V – D^1$ is non-adjacent to at least one vertex of D^1. Hence, D^1 is also an E.D.C.D set of (G + uv). As u and v are any arbitrary pair of non-adjacent vertices, $\gamma_{edcl}(G + uv) = 4$, for every pair of non-adjacent vertices u and v of G. Therefore, $\gamma_{edcl}(G + e) < \gamma_{edcl}(G)$, for every $e \notin G$.

Hence the theorem.

Remark 3.1.1: As any 5-D.C.D type (I) critical graph is 5-E.D.C.D type (I) critical, we have the following results without proof.

Theorem 3.1.3: Let G be a 5-E.D.C.D type (I) critical graph. Then we have the following results without proof.

1. G is of diameter equal to 4.
2. G can have at most two pendant vertices.
3. G can have at most 3 cut vertices. Also if u is a cut vertex of G, then

(i) G – u can have at most two components;

(ii) One of the components is a clique.

4. G is diameter edge (addition) critical.

Theorem 3.1.4: If G is a 6-D.C.D type (II) critical graph, then G is 6-E.D.C.D type (II) critical.

Proof: Let G be a 6-D.C.D type (II) critical graph and let D be a minimum D.C.D set of G. Then G is self-centered of diameter 3 and <D> is a cycle C_6. Also, every D.C.D set of G is an E.D.C.D set of G (by Theorem 6.2.4). Now, addition of an edge between a pair of non-adjacent vertices u and v in G will reduce the radius of G to 2. Therefore, $\gamma_{dcl}(G) = 4$. If D^1 is a minimum D.C.D set of (G + uv), then every vertex of V – D must be at an even distance from at least one vertex of D^1 (as every vertex of G lies on a C_6). Hence, G is 6-E.D.C.D type (II) critical.

Remark 3.1.2: As any 6-D.C.D type (II) critical graph is 6-E.D.C.D type (II) critical, we have the following results without proof.

Theorem 3.1.5: Let G be a 6-E.D.C.D type (II) critical graph. Then we have the following:

1. G is self-centered of diameter 3.

2. G is a block.

3. G is radius edge (addition) critical.

3.2 Generalization of k-E.D.C.D critical graphs

If G is a k-D.C.D critical graph (both type (I) and type (II) structures) with $k \geq 7$, then G is of radius greater than or equal to 3. Therefore, we have the following theorems without proof.

Theorem 3.2.1: Every k-D.C.D, $k \geq 5$ (k is odd) type (I) critical graph is k-E.D.C.D type (I) critical.

Theorem 3.2.2: Every k-D.C.D, $k \geq 6$ (k is even) type (II) critical graph is k-E.D.C.D type (II) critical.

Theorem 3.2.3: If G is a k-E.D.C.D, $k \geq 5$ (k is odd) type (I) critical graph, then

1. G is of diameter equal to $k - 1$.
2. G can have at most two pendant vertices.
3. G can have at most $k - 2$ cut vertices.
4. G is diameter edge (addition) critical.

Theorem 3.2.4: If G is a k-E.D.C.D, $k \geq 6$ (k is even) type (II) critical graph, then

1. G is $\frac{k}{2}$-self-centered.
2. G is a block.
3. G is radius edge (addition) critical.

Theorem 3.2.5: Every k-D.C.D, $k \geq 5$ (k is odd or even) type (III) critical graph is k-E.D.C.D type (III) critical.

Proof: Let G be a k-D.C.D type (III) critical graph. Then G has the structure given in Figure 3.9. In this structure, the central vertices of G form a clique. Also, G is of radius 2 and diameter 3. In order to prove G is also k-E.D.C.D type (III) critical, the following claims are proved.

Claim 1: ***Every minimum D.C.D set of G is a minimum E.D.C.D set of G***

Clearly, every minimum D.C.D set D of G must contain all the $(k - 2)$ central vertices and two peripheral vertices in two different cliques attached with the central vertices. Thus, every vertex in $V - D$ is a peripheral vertex and it is at distance two from exactly $(k - 3)$ central vertices of D (as every peripheral vertex is adjacent with exactly one central vertex of G). Hence, D is also a minimum E.D.C.D set of G and hence every minimum D.C.D set of G is a minimum E.D.C.D set of G.

Claim 2: ***Every minimum D.C.D set of (G + e) is a minimum E.D.C.D set of (G + e), for every additional edge e in G***

If we add an edge e between a pair of non-adjacent vertices x and y of G, then $\gamma_{dcl}(G + e) = k - 1$. Also, every minimum D.C.D set D^1 of (G + e) must contain exactly the $(k - 3)$ number of central vertices and the set {x, y}. Clearly every vertex in $V – D^1$ is at a distance two from at least one vertex of D^1. Hence, D^1 is also a minimum E.D.C.D set of (G + e) and hence every minimum D.C.D set of (G + e) is a minimum E.D.C.D set of (G + e), for every additional edge e in G.Hence the proof.

4. CONCLUSION

In this paper, the critical concept of even distance closed domination is defined and the structural properties of k-E.D.C.D critical graphs are analyzed. The structural properties of k-E.C.R. D critical graphs such as radius, diameter, number of cut vertices, clique components and Hamiltonian properties are studied. Since this concept deals the reduction in the cardinality of even distance closed dominating set for any addition of one new link in the original structure, it will be useful to study the communication model, which reduces it dominating parameters by simple addition of a link, which doesn't exist in the system. Hence this critical concept can be directly applied to the construction of a fault tolerant communication model.

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