Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

# On a Paper in Connection with the Derivation of Generating Functions Involving Laguerre Polynomials

**Biswas Nabin Chandra[1*] and ChongdarAmartya[2]**

[1*]Department of Mathematics, Dum Dum Motijheel College, Kolkata-700074, W.B., India.
[2]Meghdoot Apartment, Flat No. D3, J-G/7, Aswininagar, Kolkata-700159, West Bengal, India
Email : nabinchandra_math@yahoo.com

## ABSTRACT

In this note, we give some observations on the derivation of generating functions involving Laguerre polynomial presented by Das and Chatterjee. Finally, we have extended the results of Das and Chatterjee.

**KEYWORDS :** Laguerre polynomial, Generating functions.

***Corresponding Author :**

**Nabin Chandra Biswas**

Assistant Professor

Department of Mathematics

Dum Dum Motijheel College,

1,Moijheel Avenue, Kolkata-700074,

West Bengal, India

Email : nabinchandra_math@yahoo.com

# INTRODUCTION

Das and Chatterjee in their paper[1]claimed that the operator

$$A = xy^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - xyz^{-1},$$

such that

$$A\left[y^{\alpha}z^{n}L_{n}^{(\alpha)}(x)\right] = (n+1)L_{n+1}^{(\alpha-1)}(x)$$

……….(1.1)

and

$$e^{aA}f(x,y,z) = \exp\left(-axy^{-1}z\right)f\left(x + axy^{-1}z, y + az, z\right)$$

……….(1.2)

is new and obtained the result

$$(1+t)^{\alpha}\exp(-xt)\,L_{n}^{(\alpha)}(x+xt) = \sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{(\alpha-m)}(x)t^{m}, \quad (|t| \le 1).$$

……….(1.3)

Finally, they obtained three theorems on bilateral and mixed trilateral generating functions with the help of (1.3) and the operator $A$ given above.

N.Barik in his paper[2] proved some theorems on generating functions of Laguerre polynomials by using a linear partial differential operator. Subsequently, M.C. Mukherjee in his paper[3] made some comments in the light of the work[4].

In this section of the present note, we have given some of our observations on the works[1.2]. At first, we would like to mention that the authors of the paper[1] perhaps fail to notice the work[4] in which the above mentioned operator $A$ and the result (1.3) are found derived while investigating generating functions of Laguerre polynomial by group theoretic method due to Weisner[5-7]. It may also be noted that the theorems proved in the papers[1,2] are the direct consequences of the results obtained by the operator $A_{22}$, defined in the paper[4], which is same as $A$ in the paper[1]. This fact has been clearly shown in the paper[3] while writing a note on the work[2]. Besides, the theorems (1 and 3) in the paper[1] and the theorems (3A and 5A) in the paper[2] are same. Furthermore, the methodology described in both the papers[1,2] is not new.

In section 2, we obtain the extensions of the theorems (1-3) obtained in the paper[1] by the same technique using the operator $A$ and the relation (1.3).

Finally in section 3, we have converted the results of section 2 into mixed trilateral generating functions with Tchebycheff polynomial by using the method as given in the paper[8].

## EXENSIONS OF THE RESULTS STATED IN THE PAPER[1]

In this section we obtain the following theorems as the extensions of the theorems by Das and Chatterjee.

### *Theorem 1*

If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) t^n \qquad \text{..........(2.1)}$$

then

$$\sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x) \sigma_n(y) t^n = (1+t)^\alpha \exp(-xt)\, F\left(x(1+t), \frac{yt}{1+t}\right) \qquad \text{..........(2.2)}$$

where

$$\sigma_n(y) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} y^k.$$

### *Theorem 2*

If there exists a generating relation of the form

$$F(x,y,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) g_n(y) t^n \qquad \text{..........(2.3)}$$

where $g_n(y)$ is and arbitrary polynomial of degree $n$, then

$$\sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x) \sigma_n(y,z) t^n = (1+t)^\alpha \exp(-xt)\, F\left(x(1+t), y, \frac{zt}{1+t}\right) \qquad \text{..........(2.4)}$$

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} g_k(y) z^k.$$

### *Theorems 3*

If there exists a generating relation of the form

$$G(x,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha)}(x) t^n \qquad \text{..........(2.5)}$$

then

$$(1+t)^\alpha \exp(-xt)\, G(x(1+t), yt) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{n+r}^{(\alpha-n+k)}(x) y^k. \qquad \text{..........(2.6)}$$

### *Proof of Theorem 1*

Now,

$$L.H.S$$

$$= \sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x)\sigma_n(y)t^n$$

$$= \sum_{n=0}^{\infty} t^n L_{n+r}^{(\alpha-n)}(x) \sum_{k=0}^{n} a_k \binom{n+r}{k+r} y^k$$

$$= \sum_{n=0}^{\infty} t^{n+k} L_{n+k+r}^{(\alpha-n-k)}(x) \sum_{k=0}^{\infty} a_k \binom{n+k+r}{k+r} y^k$$

$$= \sum_{k=0}^{\infty} a_k (yt)^k \sum_{n=0}^{\infty} \binom{n+k+r}{k+r} L_{n+k+r}^{(\alpha-n-k)}(x) t^n$$

$$= \sum_{k=0}^{\infty} a_k (yt)^k (1+t)^{\alpha-k} \exp(-xt)\, L_{k+r}^{(\alpha-k)}(x(1+t)) \text{[by using (1.3)]}$$

$$= (1+t)^{\alpha} \exp(-xt) \sum_{k=0}^{\infty} a_k \left(\frac{yt}{1+t}\right)^k L_{k+r}^{(\alpha-k)}(x(1+t))$$

$$= (1+t)^{\alpha} \exp(-xt)\, F\left(x(1+t), \frac{yt}{1+t}\right) \quad \text{[by using (2.1)]}$$

$$= R.H.S.$$

This proves the Theorem 1 and is also obtained by A.K.Chongdar[9] in course of application of a general theorem on various special functions by classical method.

### *Corollary 1*

Putting $r = 0$ in Theorem 1, we get the Theorem 1 of the paper[1].

### *Proof of Theorem 2*

It is a routine work and is exactly similar to the proof of Theorem1 given above.

### *Corollary 2*

Putting $r = 0$ in Theorem 2, we get the Theorem 2 of the paper[1].

### *Proof of Theorem 3*

Now,

$$L.H.S$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{n+r}^{(\alpha-n+k)}(x) y^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \binom{n+k+r}{k+r} L_{n+k+r}^{(\alpha-n)}(x)(yt)^k t^n$$

$$= \sum_{k=0}^{\infty} a_k \left( \sum_{n=0}^{\infty} \binom{n+k+r}{k+r} L_{n+k+r}^{(\alpha-n)}(x) t^n \right) (yt)^k$$

$$= (1+t)^{\alpha} \exp(-xt) \sum_{k=0}^{\infty} a_k L_{k+r}^{(\alpha)}\big(x(1+t)\big)(yt)^k \quad \text{[by using (1.3)]}$$

$$= (1+t)^{\alpha} \exp(-xt)\, G(x(1+t), yt) \quad \text{[by using (2.5)]}$$

$$= R.H.S.$$

This proves the Theorem 3 and is also found derived in the paper[10]by classical method.

### *Corollary 3*

Putting $r = 0$ in the above theorem, we get the Theorem 3 of the paper[1].

The three theorems proved above are found derived in the paper[11] by using a partial differential operator obtained by single interpretation in Weisner's method.

## CONVERSION OF THE ABOVE BILATERAL GENERATING FUNCTION INTO TRILATERAL GENERATING FUNCTIONS WITH TCHEBYCHEFF POLYNOMIAL

In this section we shall convert the theorems (1-3) into trilateral generating functions with Tchebycheff polynomial by means of the relation

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right]$$

……….(3.1)

## Conversion to trilateral generating functions

Now to convert the bilateral generating relation stated in Theorem 1 into a trilateral generating relation with Tchebycheff polynomial we notice that

$$\sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x)\sigma_n(y) T_n(z) t^n$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(t\left(z + \sqrt{z^2-1}\right)\right)^n L_{n+r}^{(\alpha-n)}(x)\sigma_n(y) + \sum_{n=0}^{\infty} \left(t\left(z - \sqrt{z^2-1}\right)\right)^n L_{n+r}^{(\alpha-n)}(x)\sigma_n(y)\right]$$

$$= \frac{1}{2}\left[(1+\rho_1)^{\alpha} \exp(-x\rho_1)\, F\left(x(1+\rho_1), \frac{y\rho_1}{1+\rho_1}\right) + (1+\rho_2)^{\alpha} \exp(-x\rho_2)\, F\left(x(1+\rho_2), \frac{y\rho_2}{1+\rho_2}\right)\right]$$

where

$$\rho_1 = t\left(z + \sqrt{z^2 - 1}\right), \qquad \rho_2 = t\left(z - \sqrt{z^2 - 1}\right).$$

Thus we obtain the following trilateral generating theorem :

### *Theorem 4*

If

$$F(x,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) t^n$$

……….(3.2)

then

$$\sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x)\sigma_n(y)T_n(z)t^n =$$

$$= \frac{1}{2}\left[(1+\rho_1)^{\alpha}\exp(-x\rho_1)\, F\left(x(1+\rho_1), \frac{y\rho_1}{1+\rho_1}\right) + (1+\rho_2)^{\alpha}\exp(-x\rho_2)\, F\left(x(1+\rho_2), \frac{y\rho_2}{1+\rho_2}\right)\right]$$

……….(3.3)

where

$$\sigma_n(y) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} y^k,$$

$$\rho_1 = t\left(z + \sqrt{z^2 - 1}\right), \qquad \rho_2 = t\left(z - \sqrt{z^2 - 1}\right).$$

### *Corollary 4*

Putting $r = 0$ in Theorem 4, we get

If

$$F(x,t) = \sum_{n=0}^{\infty} a_n L_{n}^{(\alpha-n)}(x) t^n$$

……….(3.4)

then

$$\sum_{n=0}^{\infty} L_{n}^{(\alpha-n)}(x)\sigma_n(y)T_n(z)t^n =$$

$$= \frac{1}{2}\left[(1+\rho_1)^{\alpha}\exp(-x\rho_1)\, F\left(x(1+\rho_1), \frac{y\rho_1}{1+\rho_1}\right) + (1+\rho_2)^{\alpha}\exp(-x\rho_2)\, F\left(x(1+\rho_2), \frac{y\rho_2}{1+\rho_2}\right)\right]$$

……….(3.5)

where

$$\sigma_n(y) = \sum_{k=0}^{n} a_k \binom{n}{k} y^k,$$

$$\rho_1 = t\left(z + \sqrt{z^2 - 1}\right), \qquad \rho_2 = t\left(z - \sqrt{z^2 - 1}\right).$$

Proceeding exactly in the same way, we can convert Theorem 2 and Theorem 3 into trilateral generating relations with Tchebycheff polynomial stated in Theorem 5 and Theorem 6 below.

### *Theorem 5*

If there exists a generating relation of the form

$$F(x, y, t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha-n)}(x) g_n(y) t^n$$

……….(3.6)

where $g_n(y)$ is and arbitrary polynomial of degree $n$, then

$$\sum_{n=0}^{\infty} L_{n+r}^{(\alpha-n)}(x)\sigma_n(y,z)T_n(u)t^n$$

$$= \frac{1}{2}\left[(1+\rho_1)^{\alpha} \exp(-x\rho_1)\, F\left(x(1+\rho_1), y, \frac{z\rho_1}{1+\rho_1}\right) + (1+\rho_2)^{\alpha} \exp(-x\rho_2)\, F\left(x(1+\rho_2), y, \frac{z\rho_2}{1+\rho_2}\right)\right]$$

……….(3.7)

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} g_k(y) z^k,$$

$$\rho_1 = t\left(u + \sqrt{u^2 - 1}\right), \qquad \rho_2 = t\left(u - \sqrt{u^2 - 1}\right).$$

### *Corollary 5*

Putting $r = 0$ in Theorem 5, we get the following result

If

$$F(x, y, t) = \sum_{n=0}^{\infty} a_n L_{n}^{(\alpha-n)}(x) g_n(y) t^n$$

……….(3.8)

then

$$\sum_{n=0}^{\infty} L_{n}^{(\alpha-n)}(x)\sigma_n(y,z)T_n(u)t^n$$

$$= \frac{1}{2}\left[(1+\rho_1)^{\alpha} \exp(-x\rho_1)\, F\left(x(1+\rho_1), y, \frac{z\rho_1}{1+\rho_1}\right) + (1+\rho_2)^{\alpha} \exp(-x\rho_2)\, F\left(x(1+\rho_2), y, \frac{z\rho_2}{1+\rho_2}\right)\right]$$

……….(3.9)

where

$$\sigma_n(y,z) = \sum_{k=0}^{n} a_k \binom{n}{k} g_k(y) z^k,$$

$$\rho_1 = t\left(u + \sqrt{u^2 - 1}\right), \qquad \rho_2 = t\left(u - \sqrt{u^2 - 1}\right).$$

### *Theorem 6*

If there exists a generating relation of the form

$$G(x,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(\alpha)}(x) t^n$$

……….(3.10)

then

$$\sum_{n=0}^{\infty} \sigma_n(x,y) T_n(z) t^n =$$

$$= \frac{1}{2}[(1+\rho_1)^\alpha \exp(-x\rho_1)\, G(x(1+\rho_1), y\rho_1)$$

$$+ (1+\rho_2)^\alpha \exp(-x\rho_2)\, G(x(1+\rho_2), y\rho_2)]$$

……….(3.11)

where

$$\sigma_n(x,y) = \sum_{k=0}^{n} a_k \binom{n+r}{k+r} L_{n+r}^{(\alpha-n+k)}(x) y^k,$$

$$\rho_1 = t\left(z + \sqrt{z^2 - 1}\right), \qquad \rho_2 = t\left(z - \sqrt{z^2 - 1}\right).$$

### *Corollary 6*

Putting $r = 0$ in the above theorem, we get the following result

If

$$G(x,t) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) t^n$$

……….(3.12)

then

$$\sum_{n=0}^{\infty} \sigma_n(x,y) T_n(z) t^n =$$

$$= \frac{1}{2}[(1+\rho_1)^\alpha \exp(-x\rho_1)\, G(x(1+\rho_1), y\rho_1)$$

$$+ (1+\rho_2)^\alpha \exp(-x\rho_2)\, G(x(1+\rho_2), y\rho_2)]$$

……….(3.13)

where

$$\sigma_n(x,y) = \sum_{k=0}^{n} a_k \binom{n}{k} L_n^{(\alpha-n+k)}(x) y^k,$$

$$\rho_1 = t\left(z + \sqrt{z^2 - 1}\right), \qquad \rho_2 = t\left(z - \sqrt{z^2 - 1}\right).$$

## CONCLUSION

In conclusion, it is obvious that not only the theorems stated in the paper[1,2] but also their extensions can be easily derived by using the operator $A_{22}$ in the paper[4], which is $A$ in the paper[1]. Furthermore, by using the theorems (1-3), we canimmediately generalize any known result of the form (2.1) or (2.5) from therelations (2.2) or (2.6). Thus a large number of generating relations can beeasily obtained by attributing different suitable values to $a_n$ in (2.1) or (2.5).

## ACKNOWLEDGEMENT

We are grateful to Prof. A.K. Chongdar for his kind help in the preparation of this paper.

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