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A Result on Fixed Point in Complete G-Metric Space

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ABSTRACT

In this paper , we define a generalized weakly contraction mapping in G-metric space and present a fixed point theorem for such mapping in complete G-metric space which generalizes a result in the recent literature.

KEYWORDS: G-metric space , fixed point, self map, generalized weakly contraction mapping

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INTRODUCTION AND PRILIMINARIES

In 2006, Mustafa and Sims[2] introduced a notion of generalized metric space called G-metric space . Later, several authors established many fixed point results for self mappings in G-metric spaces under certain contractions.

Definition 1.1. [2] Let X be a non empty set and let $G: X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

$G(x, y, z) = 0$ if $x = y = z$.

$G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$.

$G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$.

$G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables)

$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangular inequality)

Then the function G is called a **G-metric on X** and the pair (X, G) is called a **G-metric space**.

Example.[2] Let (X, d) be a usual metric space. Then (X, G) is G-metric space, where $G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, for all $x, y, z \in X$.

Definition 1.2.[3] Let (X, G) and (X', G') be G-metric spaces and $f: (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if for given $\varepsilon > 0$ there exist $\delta > 0$ such that for $x, y \in X, G(a, x, y) < \delta$ implies that $G(fa, fx, fy) < \varepsilon$.

A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 1.3. [3] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$. That is,for any $\varepsilon > 0$, there exist N∈n such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq \mathrm{n}$. We call x as the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.4. [3] Let (X, G) and (X', G') be G metric spaces. A function $f: X \to X$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, $\{fx_n\}$ is G-convergent to $f(x)$ whenever $\{x_n\}$ is G-convergent to x.

Proposition 1.5.[4] Let (X, G) be a G-metric space. Then the following statements are equivalent:

$\{x_n\}$ is G-convergent to x.

$G(x_n, x_n, x) \to 0$ as $n \to \infty$.

$G(x_n, x, x) \to 0$ as $n \to \infty$.

$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Definition 1.6.[3] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if given $\varepsilon > 0$, there is N∈ $\mathbb{n}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq$N, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition1.7.[3] A G-metric space (X, G) is said to be G-complete if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.8. [4] Every G-metric space (X, G) defines a metric space (X, d_G), where d_G defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Proposition 1.9. [2] Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, the following hold.

If $G(x, y, z) = 0$ then $x = y = z$.

$G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.

$G(x, y, y) \leq 2\, G(y, x, x)$.

$G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

$G(x, y, z) \leq \frac{2}{3}\{G(x, a, a) + G(y, a, a) + G(z, a, a)\}$.

Definition 1.10. [5] A mapping $\psi: [0, \infty) \to [0, \infty)$ is called an altering distance function if ψ is continuous and non-decreasing

and $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.11. [1] A self mapping $T: X \to X$, where (X, d) is a metric space, is said to be a generalized weakly contractive mapping if

$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \emptyset(\max\{d(x, y), d(y, Ty)\})$ for all x,y where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$, ψ is an altering distance function and $\emptyset: [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

B.S.Choudhury et al. proved the following theorem in [1].

Theorem 1.12. Let (X,d) be a complete metric space and T be a generalized weakly contractive self mapping of X. Then T has a unique fixed point.

Motivated by the above result, we prove the same result in G-metric spaces.

Definition 1.13. A mapping $T: X \to X$, where (X, G) is a G-metric space, is said to be a generalized weakly contractive mapping if

$$\varphi(G(Tx, Ty, Tz) \leq \varphi(M(x, y, z)) - \emptyset(\max\{G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz)\})$$

$$for\ all\ x, y, z \in X \qquad (1.14)$$

where $M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz),$

$\frac{1}{3}\{G(x,Ty,Ty) + G(y,Tz,Tz) + G(z,Tx,Tx)\}$,

φ is an altering distance function and $\emptyset: [0,\infty) \to [0,\infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

MAIN RESULT

Theorem 2.1. A generalized weakly contractive self mapping of a complete G-metric space (X,G) has a unique fixed point.

Proof : Let $x_0 \in X$ be arbitrary . Then we can define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \geq 1$.

If $x_n = x_{n-1}$ for some n, then T has a fixed point.

So assume that $x_n \neq x_{n-1}$ for all $n \geq 1$

Now, from (1.14) we have for all $n \geq 1$

$$\varphi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \leq$$

$$\varphi(M(x_n, x_{n+1}, x_{n+1})) - \emptyset\begin{pmatrix}\max\{G(x_n, x_{n+1}, x_{n+1}), G(Tx_n, Tx_{n+1}, Tx_{n+1}), \\ G(Tx_n, Tx_{n+1}, Tx_{n+1}),\}\end{pmatrix}. \qquad (2.2)$$

where,

$$M(x_n, x_{n+1}, x_{n+1}) = \max\Big\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}),$$

$$G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \ \frac{1}{3}\big(G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) +$$

$$G(x_{n+1}, Tx_n, Tx_n)\big)\Big\} =$$

$$\max\left\{\begin{matrix} G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_{n-1}, Tx_n, Tx_n), \\ G(Tx_n, Tx_{n+1}, Tx_{n+1}), \\ \frac{1}{3}\big(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_n, Tx_n)\big)\end{matrix}\right\}$$

$$= \max\Big\{G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}), \ \frac{1}{3}\big(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1})$$

$$+ G(Tx_n, Tx_n, Tx_n)\big)\Big\}$$

But $\frac{1}{3}\big(G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Tx_n, Tx_n)\big)$

$\leq \max\{ G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$

So $M(x,y,z) = \max\{ G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$

Now from (1.14),we have

$$\varphi\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) \le \varphi(\max\{G(Tx_{n-1}, Tx_n, Tx_n),\ G(Tx_{n-1}, Tx_{n+1}, Tx_{n+1})\ ) - \emptyset(\max\{G(Tx_{n-1}, Tx_n, Tx_n), G(Tx_n, Tx_{n+1}, Tx_{n+1}),\ \}). \qquad (2.3)$$

Suppose $G(Tx_{n-1}, Tx_n, Tx_n) \le G(Tx_n, Tx_{n+1}, Tx_{n+1})$ for some integer n.

Then from (2.3), we have

$$\varphi\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) \le \varphi\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) - \emptyset\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big)$$

$\emptyset\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) \le 0$ implies

$\emptyset\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) = 0$ which gives`

$G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0$,that is, $G(x_{n+1}, x_{n+2}, x_{n+2}) = 0$

or $x_{n+2} = x_{n+1}$,a contradiction.

Hence $G(Tx_n, Tx_{n+1}, Tx_{n+1}) < G(Tx_{n-1}, Tx_n, Tx_n)$ ∀ n≥1

Thus $\{G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$ is monotonic decreasing sequence and hence is converging to some real number r ≥ 0.

i.e., $\lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = r$

Suppose r > 0.

$$\varphi\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) \le \varphi\big(G(Tx_{n-1}, Tx_n, Tx_n)\big) - \emptyset\big(G(Tx_{n-1}, Tx_n, Tx_n)\big).$$

Letting n→∞, we get $\varphi(r) \le \varphi(r) - \emptyset(r)$ implies

$\emptyset(r) \le 0$, a contradiction.

Hence r = 0.

or $\lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0$.

i.e., $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ (2.4)

We now show that the sequence $\{x_n\}$ is Cauchy.

Suppose $\{x_n\}$ is not Cauchy. Then there exists an ε > 0 and two sequences $\{m(k)\}$ and $\{n(k)\}$ with n(k) > m(k) > k for all positive integers k such that $G\big(x_{m(k)}, x_{n(k)}, x_{n(k)}\big) \ge \varepsilon$ and

$G\big(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}\big) < \varepsilon$. (2.5)

Now, by rectangular inequality, (2.4) and (2.5), we obtain

$$\varepsilon \le G\big(x_{m(k)}, x_{n(k)}, x_{n(k)}\big) \le G\big(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}\big) + G\big(x_{n(k)-1}, x_{n(k)}, x_{n(k)}\big)$$

So $\varepsilon \le G\big(x_{m(k)}, x_{n(k)}, x_{n(k)}\big) < \varepsilon + G\big(x_{m(k)-1}, x_{m(k)}, x_{n(k)}\big)$

Letting k→∞ in the above inequalities, we get

$\varepsilon \le \lim_{k\to\infty} G\big(x_{m(k)}, x_{n(k)}, x_{n(k)}\big) < \varepsilon + 0$

Therefore $\lim_{k\to\infty} G\big(x_{m(k)}, x_{n(k)}, x_{n(k)}\big) = \varepsilon$. (2.6)

Again by rectangular inequality, we have

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{m(k)+1}, x_{m(k)+1}) + G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)})$$

and

$$G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})$$

Letting k→∞, we get from (2.4) and (2.6),

$$\lim_{k\to\infty} G(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon$$

Again by rectangular inequality,

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) \quad \text{and}$$

$$G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) .$$

Letting k→∞ in the above inequalities, we get

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon.$$

Similarly, we can show that

$$\lim_{k\to\infty} G(x_{n(k)}, x_{m(k)+1}, x_{m(k)+1}) = \varepsilon.$$

Now, put $x = x_{m(k)},\ y = x_{n(k)},\ z = x_{n(k)}$ in (1.14)

$$\varphi\left(G(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)})\right) \leq \varphi(\mathrm{M}\{G(x_{m(k)}, x_{n(k)}, x_{n(k)})\})$$

$$-\emptyset\left(\max\left\{\begin{matrix} G(x_{m(k)}, x_{n(k)}, x_{n(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}), \\ G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) \end{matrix}\right\}\right) \quad (2.7)$$

where

$$M(x_{m(k)}, x_{n(k)}, x_{n(k)})$$

$$= \max\Big\{G(x_{m(k)}, x_{n(k)}, x_{n(k)}), G(x_{m(k)}, Tx_{m(k)}, Tx_{m(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}),$$

$$G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}),\ \frac{1}{3}\Big(G(x_{m(k)}, Tx_{n(k)}, Tx_{n(k)}) + G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})$$

$$+ G(x_{n(k)}, Tx_{m(k)}, Tx_{m(k)})\Big)\Big\}$$

$$= \max\Big\{G(Tx_{m(k)-1}, Tx_{n(k)-1}, Tx_{n(k)-1}), G(Tx_{m(k)-1}, Tx_{n(k)}, Tx_{n(k)}),$$

$$G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)}),\ \frac{1}{3}\Big(G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)})$$

$$+ G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)}) + G(Tx_{n(k)-1}, Tx_{n(k)}, Tx_{n(k)})\Big)\Big\}$$

Letting k→∞, and using (2.4), (2.6), we get

$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, x_{n(k)})$ =max {ε, 0, 0, $\frac{1}{3}$(ε +0+ ε)} = ε

Now , taking limit as k→∞ in (2.6), we get

$\varphi(\varepsilon) \le \varphi(\varepsilon) - \emptyset(\varepsilon)$ implies $\emptyset(\varepsilon) \le 0$, a contraction.

Thus $\{x_n\}$ is Cauchy in X. Since X is a complete G- metric space, we can find a t∈ X such that $\lim_{n\to\infty} x_n = t$.

We shall now show that t is a fixed point of T.

Put x = x_n, y = t, z = t in (1.14), we get

$$\varphi(G(Tx_n, Tt, Tt))$$
$$\le \varphi(M\{G(x_n,t,t)\}) - \emptyset(\max\{G(x_n,t,t), G(t,Tt,Tt), G(t,Tt,Tt)\})$$
$$= \varphi \max\left\{\begin{matrix} G(x_n,t,t), G(x_n,Tx_n,Tx_n), G(t,Tt,Tt), G(t,Tt,Tt), \\ \frac{1}{3}\left(G(x_n,Tt,Tt) + G(t,Tt,Tt) + G(Tt,Tx_n,Tx_n)\right) \end{matrix}\right\}$$
$$- \emptyset \max\{G(x_n,t,t), G(t,Tt,Tt) G(t,Tt,Tt)\}$$

$\varphi(G(x_{n+1}, Tt, Tt)) \le$

$$\varphi\left(\max\left\{0, G(t,Tt,Tt), \frac{2}{3}\left(G(t,Tt,Tt)\right)\right\}\right) - \emptyset \max\{G(t,t,t), G(t,Tt,Tt)\}$$

Taking limit as n→∞,

$\varphi(G(t,Tt,Tt)) \le \varphi(G(t,Tt,Tt)) - \emptyset(G(t,Tt,Tt))$

So $G(t,Tt,Tt)$= 0 which gives Tt = t. This shows that t is a fixed point of T.

To prove uniqueness, let t^1 be another fixed point of T.

Put x = t, y = t^1, z = t^1 in (1.14), we get

$\varphi(G(Tt, Tt^1, Tt^1)) \le \varphi(M(t, t^1, t^1)) - \emptyset(\max(G(t,t^1,t^1), G(t^1,Tt^1,Tt^1), G(t^1,Tt^1,Tt^1)))$

where $M(t,t^1,t^1) = \max\{G(t,t^1,t^1), G(t,Tt,Tt), G(t^1,Tt^1,Tt^1), G(t^1,Tt^1,Tt^1), \frac{1}{3}(G(t,Tt^1,Tt^1) + G(t^1,Tt^1,Tt^1) + G(t^1,Tt,Tt))\}$

$$= \max\left\{G(t,t^1,t^1), 0,0,0, \frac{1}{3}\left(G(t,t^1,t^1) + G(t^1,t^1,t^1) + G(t^1,t,t)\right)\right\}$$

$= G(t,t^1,t^1)$, since $G(t^1,t,t) \le 2G(t,t^1,t^1)$

So $\varphi(G(t,t^1,t^1)) \le \varphi(G(t,t^1,t^1)) - \emptyset(G(t,t^1,t^1))$

which implies $G(t,t^1,t^1)$ = 0 and therefore t = t^1 .

Corollary 2.8. Suppose (X, G) is a complete G-metric space.If the mapping $T: X \to X$ satisfies the following condition $\forall x, y, z \in X$ $\varphi(G(Tx, Ty, Tz) \le$

$\varphi(\max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)) -$

$\emptyset(\max\{G(x,y,z), G(y,Ty,Ty), G(z,Tz,Tz)\})$, where

φ is an altering distance function and $\emptyset: [0,\infty) \to [0,\infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

Then T has a unique fixed point in X.

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