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Result on Unique Common Fixed Point of Two Continuous Mappings

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ABSTRACT:

In this paper, generalization of common fixed point is proved under a generalized inequality involving two self-mappings. In other words Let X be a closed subspace of a Hilbert Space and$T_1, T_2 : X \to X$ be continuous mappings satisfying the given condition then $T_1 and T_2$ have unique common fixed point in X.

KEY WORDS: Common Fixed Point, Banach Space, Completeness.

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INTRODUCTION

Essentially, Fixed-point theorems provide the conditions under which maps have solution. The theory itself is a beautiful mixture of analysis (Pure and Applied), Topology and Geometry. Over the last 50 years or so the theory of fixed point has been revealed to be a very powerful and important tool in the study of non-linear phenomena. In particular, the techniques of the fixed point theory have been applied in various diverse fields such as Biology, Chemistry, Physics, Economics, Medicines and Game Theory etc.

The study of Existence & Uniqueness of Coincidence Point & Common Fixed Point of Mappings satisfying certain contractive conditions has been an interesting field of Mathematics from 1922, when Banach stated & proved his famous result (Banach Contraction Principle,[1]. Results found here are refine some of the generalization of Banach's theorem[3,4] and are generalized[2] by restricting the number of self mappings from three to two.

MAIN RESULT

Theorem: 1

Let X be a closed subspace of Hilbert Space and $T_1, T_2 : X \to X$be continuous mappings such that :

$$\|T_1x - T_2y\|^p \leq a_1 \frac{\|x-y\|^p[1 + \|x - T_2x\|^p]}{[1 + \|x-y\|^p]} + a_2[\|x - T_1x\|^p + \|y - T_2y\|^p] + a_3[\|x - T_2y\|^p + \|y - T_1x\|^p] + a_4\|x-y\|^p$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$. Then $T_1 \& T_2$ have unique common fixed point in subspace$X$.

Proof : Let x_0 be any point in the set X.Then define the sequence $\{x_n\}$ as follows:

$$x_{2n} = T_2x_{2n-1} for n = 1,2,3, \dots$$

$$x_{2n+1} = T_1x_{2n} for n = 0,1,2,3, \dots$$

Suppose that $n = 2m$for some integer m.Then

$$\|x_{n+1} - x_n\| = \|x_{2m+1} - x_{2m}\| = \|T_1x_{2m} - T_2x_{2m-1}\|$$

From the given condition we have,

$$\|x_{n+1} - x_n\|^p = \|x_{2m+1} - x_{2m}\|^p = \|T_1x_{2m} - T_2x_{2m-1}\|^p$$

$$\le a_1 \frac{\|x_{2m} - x_{2m-1}\|^p [1 + \|x_{2m} - T_1 x_{2m}\|^p]}{[1 + \|x_{2m} - x_{2m-1}\|^p]}$$
$$+ \ a_2[\|x_{2m} - T_1 x_{2m}\|^p + \|x_{2m-1} - T_2 x_{2m-1}\|^p]$$
$$+ a_3[\|x_{2m} - T_2 x_{2m-1}\|^p + \|x_{2m-1} - T_1 x_{2m}\|^p]$$
$$+ \ a_4\|x_{2m} - x_{2m-1}\|^p$$

This gives

$$[(1 - a_2 - 2^p a_3) + (1 - a_1 - a_2 - 2^p a_3)\|x_{2m} - x_{2m-1}\|^p]\|x_{2m+1} - x_{2m}\|^p$$
$$\le [(a_1 + a_2 + a_4) + (a_2 + 2^p a_3 + a_4)\|x_{2m} - x_{2m-1}\|^p]\|x_{2m} - x_{2m-1}\|^p$$

$$\therefore \|x_{2m+1} - x_{2m}\|^p \le p(m)\|x_{2m} - x_{2m-1}\|^p$$

where

$$p(m) = \frac{(a_1 + a_2 + a_4) + (a_2 + 2^p a_3 + a_4)\|x_{2m} - x_{2m-1}\|^p}{(1 - a_2 - 2^p a_3) + (1 - a_1 - a_2 - 2^p a_3)\|x_{2m} - x_{2m-1}\|^p} \ ; form = 0,1,2,3, \ldots$$

Clearly

$$p(m) < 1, \qquad \forall m \ge 0 \ \ as a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$$

Continuing in this way one gets

$$\|x_{n+1} - x_n\|^p = \ \|x_{2m+1} - x_{2m}\|^p$$
$$\le p(m)\|x_{2m} - x_{2m-1}\|^p$$
$$\vdots$$
$$\le p(m)^n\|x_1 - x_0\|^p$$

By Similar way one can see that above inequality is also true if n is an odd integer. Since $0 \le p(m) < 1$, the sequence $\{x_n\}$ is Cauchy sequence and therefore by completeness of X, one find $\mu \in X$ such that

$$\lim_{n\to\infty} x_n = \mu$$

Since $\{x_{2n}\} and \ \{x_{2n+1}\}$ are sub-sequences of $\{x_n\}$ one gets

$$\lim_{n\to\infty} x_{2n} = \mu = \lim_{n\to\infty} x_{2n+1}$$

Next since $T_1 \& T_2$ are continuous one arrives at

$$T_1(\mu) = T_1\left(\lim_{n\to\infty} x_{2n}\right) = \lim_{n\to\infty} T_1 x_{2n} = \lim_{n\to\infty} x_{2n+1} = \mu$$

$$T_2(\mu) = T_2\left(\lim_{n\to\infty} x_{2n-1}\right) = \lim_{n\to\infty} T_2 x_{2n-1} = \lim_{n\to\infty} x_{2n} = \mu$$

Hence μ is common fixed point of $T_1 \& T_2$. Now to prove the uniqueness of common fixed point, let us take $\nu(\mu \neq \nu) \in X$ to be another common fixed point of $T_1 \& T_2$.

While $\|\mu - \nu\| \neq 0$.

Hence it follows that

$$\|\mu - \nu\|^p = \|T_1\mu - T_2\nu\|^p$$
$$\leq a_1 \frac{\|\mu - \nu\|^p [1 + \|\mu - T_1\mu\|^p]}{[1 + \|\mu - \nu\|^p]}$$
$$+ a_2[\|\mu - T_1\mu\|^p + \|\nu - T_2\nu\|^p]$$
$$+ a_3[\|\mu - T_2\nu\|^p + \|\nu - T_1\mu\|^p]$$
$$+ a_4\|\mu - \nu\|^p$$

$$\therefore \|\mu - \nu\|^p \leq (a_1 + 2a_3 + a_4)\|\mu - \nu\|^p$$

which is a contradiction as

$$a_1 + 2a_3 + a_4 < a_1 + 2a_2 + 2^{p+1}a_3 + a_4 < 1$$

Thus $\mu = \nu$.

Theorem : 2

Let X be a closed subspace of Hilbert Space and $T_1, T_2 : X \to X$be continuous mappings such that :

$$\|T_1x - T_2y\|^p \leq a_1 \frac{\|x - y\|^p [1 + \|y - T_2y\|^p]}{[1 + \|x - T_1x\|^p]}$$
$$+ a_2 \frac{\|y - T_2y\|^p [1 + \|x - y\|^p]}{[1 + \|x - T_1x\|^p]}$$
$$+ a_3[\|y - T_2y\|^p + \|x - y\|^p]$$
$$+ a_4\|x - y\|^p$$

for all $x, y \in X$ with $x \neq y$ and $p \in \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, where a_1, a_2, a_3, a_4 are non-negative real numbers with $a_1 + a_2 + 2a_3 + a_4 < 1$.Then $T_1 \& T_2$ have unique common fixed point in subspace$X$.

Proof : Let x_0 be anypoint in the set$X$.Thendefine the sequence $\{x_n\}$ as follows :

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$$\|x_{n+1} - x_n\| = \|x_{2m+1} - x_{2m}\| = \|T_1x_{2m} - T_2x_{2m-1}\|$$

From the given condition we have,

$$\|x_{n+1} - x_n\|^p = \|x_{2m+1} - x_{2m}\|^p$$
$$= \|T_1x_{2m} - T_2x_{2m-1}\|^p$$

$$\leq a_1 \frac{\|x_{2m} - x_{2m-1}\|^p [1 + \|x_{2m-1} - T_2 x_{2m-1}\|^p]}{[1 + \|x_{2m} - T_1 x_{2m}\|^p]}$$
$$+ \ a_2 \frac{\|x_{2m-1} - T_2 x_{2m-1}\|^p [1 + \|x_{2m} - x_{2m-1}\|^p]}{[1 + \|x_{2m} - T_1 x_{2m}\|^p]}$$
$$+ \ a_3 [\|x_{2m-1} - T_2 x_{2m-1}\|^p + \|x_{2m} - x_{2m-1}\|^p]$$
$$+ \ a_4 \|x_{2m} - x_{2m-1}\|^p$$

This gives

$$[\,1 + (1 - 2a_3 - a_4)\|x_{2m} - x_{2m-1}\|^p]\|x_{2m+1} - x_{2m}\|^p$$
$$\leq [(a_1 + a_2 + 2a_3 + a_4) + (a_1 + a_2)\|x_{2m} - x_{2m-1}\|^p]\|x_{2m} - x_{2m-1}\|^p$$

$$\therefore \|x_{2m+1} - x_{2m}\|^p \leq p(m)\|x_{2m} - x_{2m-1}\|^p$$

where

$$p(m) = \frac{(a_1 + a_2 + 2a_3 + a_4) + (a_1 + a_2)\|x_{2m} - x_{2m-1}\|^p}{1 + (1 - 2a_3 - a_4)\|x_{2m} - x_{2m-1}\|^p} for\ m = 0,1,2,3, \ldots$$

Clearly

$$p(m) < 1, \qquad \forall m \geq 0 \ as a_1 + a_2 + 2a_3 + a_4 < 1$$

Continuing in this way one gets

$$\|x_{n+1} - x_n\|^p = \ \|x_{2m+1} - x_{2m}\|^p$$
$$\leq p(m)\|x_{2m} - x_{2m-1}\|^p$$
$$\vdots$$
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$$T_2(\mu) = T_2\left(\lim_{n\to\infty} x_{2n-1}\right) = \lim_{n\to\infty} T_2 x_{2n-1} = \lim_{n\to\infty} x_{2n} = \mu$$

Hence μ is common fixed point of $T_1 \& T_2$. Now to prove the uniqueness of common fixed point, let us take $\nu(\mu \neq \nu) \in X$ to be another common fixed point of $T_1 \& T_2$.

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Hence it follows that

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$$\leq a_1 \frac{\|\mu - \nu\|^p[1 + \|\nu - T_2\nu\|^p]}{[1 + \|\mu - T_1\mu\|^p]}$$
$$+ a_2 \frac{\|\nu - T_2\nu\|^p[1 + \|\mu - \nu\|^p]}{[1 + \|\mu - T_1\mu\|^p]}$$
$$+ a_3 [\, \|\nu - T_2\nu\|^p + \|\mu - \nu\|^p \,]$$
$$+ a_4\|\mu - \nu\|^p$$

$$\therefore \|\mu - \nu\|^p \leq (a_1 + a_3 + a_4)\|\mu - \nu\|^p$$

which is a contradiction as

$$a_1 + a_3 + a_4 < a_1 + a_2 + 2a_3 + a_4 < 1$$

Thus$\mu = \nu$.

CONCLUSIONS

The method adopted in the proof of common fixed point theorems reveal that yet there are various directions in which the Banach's fixed point theorem can be refined and extended.

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