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Some Characterizations of Strictly Convex 2- Normed Spaces

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ABSTRACT:

The concept 2 normed space was introduced by S. Gahler. After Daminni and A.White introduced the ideas of strictly 2 convex 2 –normed space. They gave several important result on strictly convex 2-normmed spaces. This paper consist of several characterization of strictly convex 2 normed Spaces.

KEYWORDS:

2 normed space, Strictly convex, 2 dimensional analog.

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INTRODUCTION:

Let L , be a linear space of dimension greater than L and $\|\cdot\|$ a real valued function on L which satisfies the following four conditions:

1. $\|a, b\| = 0$ if and only if a and b are linearly dependent,
2. $\|a, b\| = \|b, a\|$,
3. $\|a, \beta b\| = |\beta| \|a, b\|$, when β is real,
4. $\|a, b+c\| \leq \|a, b\| + \|a, c\|$.

$\|\cdot\|$ is called a 2- norm on L and $(L, \|\cdot\|)$ linear 2- normed space.

The 2- norm is a non-negative function.

With respect to the definition of the 2-norm the notion of Linear 2- normed space is a 2- dimensional analog to the notion of normed linear space. A normed linear space is called strictly convex if $\|x+y\| = \|x\| + \|y\|$ and $\|x\| = \|y\| = 1$ imply that $x = y$.

For non-zero vectors $a, b \in L$, let $V(a, b)$ denote the subspace of L generated by a and b . whenever the notation $V(a, b)$ is used, it will be understood that a and b are non-zero vectors. A linear 2 – normed space $(L, \|\cdot\|)$ said to be strictly convex if $\|a+b, c\| = \|b, c\|, \|a, c\| = \|b, c\| = 1$ and $c \notin V(a, b)$ imply that $a = b$.

If c is a fixed non-zero element of L , let $V(c)$ denote the linear subspace of L generated c and let L_c be the quotient space $L/V(c)$. For $a \in L$, let a_c represent the equivalence class of a with respect to $V(c)$. L_c is a vector space with addition given by $a_c + b_c = (a+b)_c$ and scalar multiplication by $\alpha a_c = (\alpha a)_c$. For arbitrary $a, b \in L$ which satisfy $a_c = b_c$, the conditions.

$\|a, c\| - \|b, c\| \|a - b, c\| = 0$, and thus, $\|a, c\| = \|b, c\|$. Therefore, the real-valued functions $\|\cdot\|_c$ on L_c given by $\|a_c\|_c = \|a, c\|$ is well-normed.

Lemma:

$\|\cdot\|_c$ is norm on L_c .

Proof:

1. $\|a_c\|_c = 0$ if and only if $\|a, c\| = 0$ i.e., if and only if $a_c = 0_c$
2. $\|\alpha a_c\|_c = \|(\alpha a)_c\|_c = \|\alpha, a, c\| = |\alpha| \|a, c\| = |\alpha| \|a, c\| = |\alpha| \|a, c\| = |\alpha| \|a, c\|$, when α is real.

$$3. \quad \|a_c + b_c\|_c = \|(a+b)_c\|_c = \|a+b, c\| \leq \|a, c\| + \|b, c\| = \|a_c\|_c + \|b_c\|_c .$$

Theorem :

For a linear 2-normed space $(L, \|\cdot\|)$, the following are equivalent :

1. $(L, \|\cdot\|)$ is strictly convex.
2. For every non-zero $c \in L$, $(L_c, \|\cdot\|)$ is strictly convex.
3. $\|a+b, c\| = \|a, c\| + \|b, c\|$ and $c \in V(a, b)$ imply that $a = \alpha a$, for some $\alpha > 0$.
4. $\|a-d, c\| = \alpha \|a-b, c\|, \|b-d, c\| = (1-\alpha) \|a-b, c\|$ $\alpha \in (0,1)$, and $c \in V(a-d, b-d)$ imply that $d = (1-\alpha)a + \alpha b$.

Proof:

1. Let $(L, \|\cdot\|)$ be strictly convex and let c be a fixed non-zero element of L . If $\|a_c + b_c\|_c = \|a_c\|_c + \|b_c\|_c$ and $\|a_c\|_c = \|b_c\|_c = 1$, then $\|a+b, c\| = \|a, c\| + \|b, c\|$ and $\|a, c\| = \|b, c\| = 1$. For the case $c \in V(a, b)$ and α and β then $0_c + c_c = \alpha a_c + \beta b_c$ or $\alpha a = -\beta b_c$. From $\|a_c\|_c = \|b_c\|_c = 1$, it follows that $\alpha = \beta$. If $\alpha = \beta$ then $c = \alpha(a+b)$ which contradicts $\|a+b, c\| = \|a_c + b_c\|_c = 2$. Therefore, $\alpha = -\beta$ and hence $a_c = b_c$. Thus $(L_c, \|\cdot\|)$ is strictly convex.
2. Assume condition 2 holds. For $c \in V(a, b)$, let $\|a+b, c\| = \|a, c\| + \|b, c\|$, i.e., $\|a_c + b_c\|_c = \|a_c\|_c + \|b_c\|_c$. By the strict convexity of $(L_c, \|\cdot\|)$, it follows that $b_c = \alpha a_c$, some $\alpha > 0$. Finally, $c \in V(a, b)$ implies that $b = \alpha a$ and condition 3 is satisfied.
3. Assume condition 3. From $\|a-d, c\| = \alpha \|a-b, c\| = 1-\alpha \|a-b, c\|$, $c \in V(a-d, b-d)$ and $\alpha \in (0,1)$, it follows that $(1-\alpha) \|a-b, c\| = \|b-d, c\| = \beta \|a-d, c\| = \alpha \beta \|a-b, c\|$ implies that $d = (1-\alpha)\alpha a + b$. Therefore, condition 4 is established.
4. Finally, assume condition 4 and let $\|a+b, c\| = \|a, c\| + \|b, c\|, \|a, c\| = \|b, c\| = 1$, where $c \in V(a, b)$. By condition 4, $0 = a-b$, i.e. $a = b$. Therefore $(L, \|\cdot\|)$ is strictly convex and the theorem is proven.

If L is a linear space of dimension greater than 1, let B'_1 be the set of all formal expression $\sum a_i \times b_i$, where a_i, b_i ($i = 1, \dots, m$) vectors in L . Let \sim be the equivalent relation on B'_1 defined by

$$\sum_{i=1}^m a_i \times b_i = \sum_{i=1}^m a_i \times b_i$$

if for arbitrary linear functions f and g on L ,

$$\sum_{i=1}^m \begin{pmatrix} f(a_i) & g(a_i) \\ f(b_i) & g(b_i) \end{pmatrix} = \sum_{i=1}^m \begin{pmatrix} f(a_i^1) & g(a_i^1) \\ f(b_i^1) & g(b_i^1) \end{pmatrix}$$

Let B_1 be the quotient space $B_1/-$. The elements of B_1 are called bivectors over L and the elements of B_1 belonging to a bivector are called representatives of this bivector. The bivector with the representative

$\sum_{i=1}^m a_i \times b_i$ will also be denoted by $\partial \left(\sum_{i=1}^m a_i \times b_i \right)$. If a bivector has a representative of the form

$\sum_{i=1}^m a_i \times b_i = a_i \times b_i$, then it is said to be simple. Only in the case where L has dimension less than or

equal to 3 does every bivector over L turn out to be simple. The space B_1 is a linear space with

$$\left(\sum_{i=1}^m a_i \times b_i \right) + \left(\sum_{i=1}^m a_{i+m} \times b_{i+m} \right) = \left(\sum_{i=1}^{m+n} a_i \times b_i \right) \text{ and } \beta \partial \left(\sum_{i=1}^m a_i \times b_i \right) = \partial \left(\sum_{i=1}^m a_i \times \beta b_i \right), \text{ when } \beta \text{ is real.}$$

If $\|\cdot\|$ is a norm on B_L , then $\|a, b\| = \|(a \times b)\|$ defines a 2-norm on L . There is an example which shows that for every 2-norm $\|\cdot\|$ on L , there need not exist a norm $\|\cdot\|$ on B_1 which satisfies $\|(a \times b)\| = \|a, b\|$ for all $a, b \in L$ if bivector L , i.e., if L has dimension less than or equal to 3, then for every 2-norm $\|\cdot\|$ on L there is a norm $\|\cdot\|$ on B_1 with $\|(a \times b)\| = \|a, b\|$ for all $a, b \in L$.

Theorem :

Let L be a linear space of dimension greater than 1, $\|\cdot\|$, be a norm on B_1 , and $\|\cdot\|$ be a 2-norm on L with $\|(a \times b)\| = \|a, b\|$ for all $a, b \in L$. If $(B_1, \|\cdot\|)$ is strictly convex, then $(L, \|\cdot\|)$ is strictly convex. If the dimension of L is less than or equal to 3 and $(L, \|\cdot\|)$ is strictly convex, then $(B_1, \|\cdot\|)$ is strictly convex.

Proof:

- Suppose $(B_1, \|\cdot\|)$ is strictly convex 1 or $c \in V(a, b)$, assume $\|a + b, c\| = \|a, c\| + \|b, c\|$ and $\|a, c\| = \|b, c\| = 1$. Then, $\|(a \times c) + (b \times c)\| = \|(a \times c)\| + \|(b \times c)\|$ and $\|*(a \times c)\| = \|*(b \times c)\| = 1$.

Since $(B_1, \|\cdot\|)$ is strictly convex, it follows that $\|(a \times c)\| = \|(b \times c)\|$. This implies that $a - b \in V(c)$. Therefore, $a = b$, since $c \notin V(a, b)$.

2. Suppose L has dimension less than or equal to 3 and $(L, \|\cdot\|)$ is strictly convex. Let $\|*1 + *2\| = \|*1\| + \|*2\|$ and $\|*1\| = \|*2\| = 1$. Hence, there exist vectors $a, b, c \in L$ with $*1 = *(a \times c)$ and $*2 = *(b \times c)$. Thus, $\|a + b, c\| = \|a, c\| + \|b, c\|$ and $\|a, c\| = \|b, c\| = 1$. If $c \in V(a, b)$, these equations imply that $c = \alpha(a - b)$ for some real, and hence $*1 = *2$. If $c \notin V(a, b)$, then strict convexity of $(L, \|\cdot\|)$ implies that $a = b$, i.e. $*1 = *2$.

If L is a 2-dimensional linear space, then B_1 is a 1-dimensional normal linear space and every 1-dimensional normed space is trivially strictly convex. Therefore, every 2-dimensional linear 2-normed space is strictly convex. Also, it follows that there are linear 2-normed space which are not strictly convex.

Theorem :

Let $(L, \|\cdot\|)$ be a linear 2-normed space and $(L', \|\cdot\|)$ be a linear normed space.

Proof:

1. If $(L, \|\cdot\|)$ is strictly convex, c is a fixed non-zero element of L , and f is a function from L into L which satisfies $\|f(a) - f(b), c\| = \|a - b\|$ for every $a, b \in L$, then the function g_c from L' into L_c , defined by $g_c(a) = [f(a)]_c$, is linear.

If $(L, \|\cdot\|)$ is strictly convex, c is a fixed non-zero element of L , and f a function from L into L' satisfying $\|f(a) - f(b)\| = \|a - b, c\|$, for every $a, b \in L$, then the function g from L into L' , defined by $g(a) = f(a) - f(0)$, is linear.

Let $(L, \|\cdot\|)$ be strictly convex and f, c, g_c be as given in statement 1. By a known theorem, $(L_c, \|\cdot\|_c)$ is strictly convex. For every $a, b \in L'$,

$$\begin{aligned} \|g_c(a) - g_c(b)\|_c &= \|[f(a) - f(b)]_c\|_c \\ &= \|f(a) - f(b), c\| \\ &= \|a - b\|. \end{aligned}$$

It follows that g_c is linear.

2. Let $(L', \|\cdot\|)$ be strictly convex and f, c, g be as given in statement 2. Then $g(0) = 0$ and for every $a, b \in L$.

$$\|g_c(a) - g_c(b)\| = \|f(a) - f(b)\| = \|a - b, c\|$$

Thus, for $a, b \in L$ which satisfy $a_c - b_c$, it follows that $\|g(a) - g(b)\| = 0$ and the function g_c from L_c into L' , given by $g_c(a_c) - g(a)$ is well-defined for any

$$\begin{aligned} a, b \in L_c \implies \|g_c(a_c) - g_c(b_c)\| &= \|g(a) - g(b)\| \\ &= \|a - b, c\| \\ &= \|a_c - b_c\|_c \end{aligned}$$

Since $g_c(0_c) = g(0) = 0$, it implies that g is linear. For any $a, b \in L$ and any real number α , $g(\alpha a) = \alpha g(a)$, and $g(a + b) = g_c(a_c + b_c) = g_c(a_c) + g_c(b_c) = g(a) + g(b)$

Therefore, g is linear.

COROLLARY:

Let $(L, \|\cdot\|)$ be a strictly convex linear 2-normed space, c be a fixed non-zero element of L , and f be a function from L into L which satisfies $f(0) = 0$ and $\|f(a) - f(b), c\| = \|a - b, c\|$ for every $a, b \in L$. Then the function g_c from L into L_c , defined by $g_c(a) = [f(a)]_c$, is linear.

Proof:

Since $(L, \|\cdot\|)$ is strictly convex it implies that $(L_c, \|\cdot\|_c)$ is strictly convex. For any $a, b \in L$,

$$\begin{aligned} \|g_c(a) - g_c(b)\|_c &= \|[f(a) - f(b)]_c\|_c \\ &= \|f(a) - f(b), c\| \\ &= \|a - b, c\| \end{aligned}$$

From part 2 of the preceding theorem, it follows that g_c is linear.

Definition:

If M and N are linear subspaces of L , a bilinear form F on $M \times N$ is said to be bounded if there is a number $K > 0$ for which $|F(a, b)| \leq K \|a, b\|$ for every $(a, b) \in M \times N$.

The norm of F , $\|F\|$ is defined by

$$\|F\| = \inf \{K : |F(a, b)| \leq K \|a, b\| \text{ for every } (a, b) \in M \times N\}$$

Theorem:

The following are equivalent :

A. $(L, \|\cdot\|)$ is strictly convex.

B. If $c \neq 0$, F is a non-zero bounded bilinear form on $L \times V(c)$, $\|x, c\| = \|y, c\| = 1$ and

$$F(x, c) = F(y, c) = \|F\|, \text{ then either } x = y \text{ or } \|x, y\| \neq 0 \text{ and } c = \pm \frac{1}{\|x, y\|^{(x-y)}}$$

Proof:

A. Assume $(L, \|\cdot\|)$ is strictly convex. Let $c \neq 0$ and F be a non-zero bounded bilinear form on $L \times V(c)$. If

$$F(x, c) = F(y, c) = \|F\| \quad \text{and} \quad \|x, c\| = \|y, c\| = 1, \quad \text{then} \quad 2 = \frac{1}{\|F\|} F(x + y, c) \leq \|x + y, c\|$$

$\leq \|x, c\| + \|y, c\| = 2$. Therefore $\|x + y, c\| = 2$. If $x \neq y$, then $c \in V(x, y)$ since otherwise the strict convexity of L would yield $x = y$. Hence, there are real numbers α and β for which $c = \alpha x + \beta y$.

Then, $1 = \|x, c\| = \|\alpha x + \beta y\| = |\beta| \|x, y\|$. Similarly $|\alpha| \|x, y\| = 1$.

Therefore, $\|x, y\| \neq 0$ and $|\alpha| = |\beta| = \frac{1}{\|x, y\|}$. Since $\|x + y, c\| = 2$, it follows that $c = \pm \frac{1}{\|x, y\|^{(x-y)}}$

B. Assume condition 2 holds and let $\|a, c\| = \|b, c\| = 1, a \neq b$ and $c \notin V(a, b)$.

Then, $\|a + b, c\| \leq \|a, c\| + \|b, c\| = 2$. If $\|a + b, c\| = 2$, there is a bounded bilinear form F

defined on $L \times V(c)$ such that $\|F\| = 1$ and $F\left(\frac{a+b}{2}, c\right) = \left\|\frac{a+b}{2}, c\right\| = 1$

Note that $F(a, c) \leq F(a, c) \leq \|F\| \|a, c\| = 1$. If $F(a, c) = 1$, then since $a \neq \frac{a+b}{2}$, condition 2

with $x = a$ and $y = \frac{a+b}{2}$ implies that $c \in V(a, b)$ which is impossible.

Thus $F(a, c) < 1$. A similar argument shows that $F(b, c) < 1$ also. Therefore,

$$1 = \frac{1}{2} F(a + b, c) = \frac{1}{2} F(a, c) + \frac{1}{2} F(b, c) < 1$$

Hence, $\|a + b, c\| < 2$ and $(L, \|\cdot\|)$ is strictly convex.

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