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Preclosed Graph Via Nets and Filter Bases

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ABSTRACT

Preclosed graph is a generalised notion than that of closed graph .Some deeper properties of this generalised closed graph have been investigated through nets and filters in this paper.

KEY WORDS. PCP (f ; x), pcl(A),PCP^{-1}(f(y)).

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1.INTRODUCTiON.

Preclosed graph was defined by Bandyopadhyay et al.[1] with the aid of preopen sets given in 1982, by Mashhour et al.[2]. Cluster set concept is a useful technique for the study of closed graph. Hamlett et. al.[3] have had recourse to this technique while they have carried out their investigations concerning closed graphs. Inspired by them attempts have been made to study preclosed graph with the aid of a set which is analogous to the cluster point set. Some basic properties of preclosed graph have been studied in this paper via nets and filters.

2.MATERIALS AND METHODS.

Throughout the paper (X, τ) or X always denotes a non trivial topological space. The following definition and proposition will be required for the presentation of the paper.

Definition 2.1.A subset A of X is called a preopen[2] set briefly a p.o. set iff $A \subset Int(Cl(A))$ .The family of all preopen sets of X is denoted by PO(X) and the family of all preopen sets containing a point $x \in X$ is denoted by PO(X,x).The complement of a p.o. set is called preclosed. The family of all preclosed subsets of X is denoted by PC(X).

Definition 2.2. The preclosure [2] of $A \subset X$ is denoted by pcl (A) and is defined by $pcl(A) = \cap \{B : B$ is preclosed and $B \supset A\}$.

Definition 2.3. For a function $f : X \to Y$, the graph G(f) is said to be preclosed [1] if for each $(x, y) \in X \times Y - G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $[U \times V] \cap G(f) = \phi$.

Proposition2.1. The function $f : X \to Y$ has a preclosed graph [1] iff for each $(x, y) \in X \times Y - G(f)$, there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $f[U] \cap V = \phi$.

Definition2.4. Let $A \subset X$, $x \in X$. Then A is a pre-neighbourhood [2] (briefly pre-nbd.) of x if there exists a $U \in PO(X, x)$ such that $x \in U \subset A$.The family of all pre-nbds of a point $x \in X$ is denoted by $N_p(x)$.

Definition 2.5. Let $f : X \to Y$, $x \in X$. The cluster set [3] of f at x, denoted by $C(f ; x)$, is defined as the set of all points y in Y such that there exists a net $<x_\alpha>$ $\alpha \in \Lambda$ in X with $x_\alpha \to x$ and $f(x_\alpha) \to y$.

Definition 2.6. Let X be a topological space and $<x_\alpha>$ $\alpha \in \Lambda$ be a net in X. Then $<x_\alpha>$ $\alpha \in \Lambda$ is said to be pre convergent [4] to a point $x \in X$, denoted by $x_\alpha \to (p)\ x$ iff $<x_\alpha>$ $\alpha \in \Lambda$ is eventually in every $V \in PO(X, x)$.

Definition 2.7. A space (X, τ) will be said to have the property P [5] if the closure is preserved under finite intersection or equivalently, the closure of intersection of any two subsets equals the intersection of their closures.

Definition 2.8. A mapping $f : X \to Y$ is called precontinuous [2] briefly pc iff for each $V \in \sigma$, $f^{-1}[V] \in PO(X)$.

Definition 2.9 . A space X is called precompact[6] if every p.o. cover of X admits a finite subcover.

Definition 2.10 . A space X is called pre-regular [7] if for each $F \in PC(X)$ and each $x \notin F$ there exist disjoint p.o. sets U and V such that $x \in U$ and $F \subset V$.

3. RESULTS AND DISCUSSIONS.

Definition 3.1. Let $f : X \to Y$, $x \in X$. The precluster set of f at x, denoted by PCP (f ; x) is defined as the set of all points y in Y such that there exists a net $<x_\alpha>, \alpha \in \Lambda$ in X with $x_\alpha \to (p) x$ and $f(x_\alpha) \to (p)\, y$.

Remark 3.1. Evidently every pre-cluster point of a function is a cluster point but the converse is not true as shown by the following example.

Example 3.1. Let $X = [-1, 1]$ and τ be the cofinite topology on X. We take the set of natural numbers N to be the directed set and let $S : N \to N$ be the net defined by $S(n) = x_n = 1/n$ for all $n \in N$. It is easy to verify that the net $< 1/n > \to 0$ but $< 1/n >$ does not preconverges to 0. Let $i : X \to X$, be the identity map. Then clearly $0 \in C(i ; 0)$ but $0 \notin PCP(i ; 0)$.

Definition 3. 2. A filter on a space X is said to preconverge to x (written $F \to (p)\, x$) iff $N_p(x) \subset F$,that is iff F is finer than the pre-nbd. filter at x.

Theorem 3.1. Let (X, τ) be a topological space and $A \subset X$. If $x \in X$, then $x \in pcl(A)$ iff there exists a net in A which preconverges to x.

Proof : Let $x \in pcl(A)$.Then every pre-nbd of x intersects A. Let $N_p(x)$ be the collection of all pre-nbds of x. Since X enjoys the property P, $(N_p(x), \subset)$ is a directed set. Now, $N \cap A \neq \phi \ \forall N \in N_p(x)$. Let $x_N \in N \cap A$. Consider the mapping $S : N_p(x) \to (p)A$ defined by $S(N) = x_N \ \forall N \in N_p(x)$. Evidently, S is a net in A and it can be seen that $S \to (p)\, x$. Sufficient part is readily obtained from the classical technique.

Theorem 3.2. Let $f : X \to Y$. Then G (f) is preclosed iff $PCP(f ; x) = \{f(x)\} \ \forall x \in X$.

Proof : Let G (f) be preclosed and $y \in PCP(f ; x)$. Then there exists a net $<x_\alpha>\ \alpha \in \Lambda$ in X with $x_\alpha \to (p)\, x$, $f(x_\alpha) \to (p) y$ and $(x_\alpha, f(x_\alpha)) \to (p)\, (x, y)$.So, $(x, y) \in pcl(G(f))$. Hence, $PCP(f ; x) = \{f(x)\} \ \forall x \in X$. Conversely, suppose $PCP(f ; x) = \{f(x)\}$ for $x \in X$ and $(x, y) \in pcl(G(f))$. So, there exists a net $<x_\alpha ; f(x_\alpha)>$ on G (f) such that $x_\alpha \to (p)\, x$ and $f(x_\alpha) \to (p) y$.Consequently, $y \in PCP(f ; x)$, from which it follows that $y = f(x)$.Therefore,$(x, y) \in G(f)$, which implies $pcl(G(f)) \subset G(f)$. So, G(f) is preclosed.

D efinition 3.3. Let F $= \{A_\alpha : \alpha \in \Lambda\}$ be a filterbase in X. Then F pre accumulates at $a \in X$ (written F ∋ a) if for every $U \in PO(X, a)$, $U \cap A_\alpha \neq \phi \quad \forall \alpha \in \Lambda$.

Theorem 3.3. Let $f : X \to Y$, $x \in X$. Then the following are equivalent :

(1) $y \in PCP(f ; x)$;

(2) $y \in \cap \{pcl(f[U]) : U$ is a pre-nbd of x};

(3) $f[N_p(x)]$ ∋ y, where $N_p(x)$ is the family of all pre-nbds of x;

(4) $f^{-1}[N_p(y)]$ ∋ x;

(5) $x \in \cap\{pcl(f^{-1}[V] : V \in N_p(y)\}$ where $N_p(y)$ is the family of all pre-nbds of y;

(6) There exists a filter F with $F \to (p)\ x$ such that $f(F) \to (p)y$.

Proof : (1) → (2). Let $y \in PCP(f ; x)$. This implies that there exists a net $x_\alpha \to (p)\ x$ such that $f(x_\alpha) \to (p)\ y$. Suppose $U \in N_p(x)$ and $V \in N_p(y)$. Hence x_α is eventually in U and $f(x_\alpha)$ is so in V. From this it follows that $f[U] \cap V \neq \phi$.Hence $y \in pcl(f[U]) \quad \forall U \in N_p(x)$ .Consequently, $y \in \cap \{pcl(f[U]) : U \in N_p(x)\}$.

(2) → (3).Suppose $y \in \cap \{pcl(f[U]) : U \in N_p(x)\}$.This indicates that $y \in pcl(f[U]) \ \forall U \in N_p(x)$.So, $f[U] \cap V \neq \phi \quad \forall V \in N_p(y)$.Thus, $\{f[U] : U \in N_p(x)\}$ ∋ y ⇒ $f[N_p(x)]$ ∋ y.

(3) → (4). Take $U \in N_p(x)$, $V \in N_p(y)$. By (3), $f[N_p(x)]$ ∋ y whence $f[U] \cap V \neq \phi \ \forall U \in N_p(x)$ and $\forall V \in N_p(y)$. It is clear that $f[X] \cap V \neq \phi \quad \forall V \in N_p(y)$ so that $f^{-1}[N_p(y)]$ is a filter base on X. From this we observe that $U \cap f^{-1}[V] \neq \phi \ \forall U \in N_p(x)$ and $\forall f^{-1}[V] \in f^{-1}[N_p(y)]$ which induces that $f^{-1}[N_p(y)]$ ∋ x.

(4) → (5). Assume $f^{-1}[N_p(y)]$ ∋ x. So, $U \cap f^{-1}[V] \neq \phi \ \forall U \in N_p(x), \forall f^{-1}[V] \in f^{-1}[N_p(y)]$, $V \in N_p(y)$.This assures that $x \in pcl(f^{-1}[V])$, $V \in N_p(y)$. Since V is arbitrary $x \in \cap \{pcl(f^{-1}[V]) : V \in N_p(y)\}$.

(1) → (6). Suppose $y \in PCP(f ; x)$. Then there exist a net $x_\alpha \to x$ such that $f(x_\alpha) \to (p)\ y$. Since $x_\alpha \to (p)\ x$, the filter F generated by the net $<x_\alpha>$ is such that $F \to (p)\ x$. Also $\{f(x_\alpha) : \alpha \in \Lambda\}$ generates a filter $f(F)$ with $f(F) \to (p)\ y$. Thus there exists a filter $F \to (p)\ x$ such that $f(F) \to (p)\ y$.

(6) → (1).Let $F \to x$ such that $f(F) \to (p)\ y$.Then the net $S = \{x_\alpha : \alpha \in D\}$ based on the filter F pre converges to x and the net $f(S) = \{f(x_\alpha) : \alpha \in D\}$ based on $f(F)$ preconverges to y.

Theorem 3.4. If $f : X \to Y$ and A be a precompact subset relative to X, then $PCP(f ; A) = \{PCP(f ; a) : a \in A\} \in PC(X)$.

Proof : Let $y \in PCP (f ; A)$.Then $y \in PCP (f ; a)$ for some $a \in A$. Then $y \in \cap \{pcl (f [U]) : U \in N_p (a)\}$.Let U be a p.o. set such that $U \supset A$. This shows that $PCP (f ; A) \subset \cap \{pcl (f [U]) : U \in PO (X), U \supset A\}$.To establish the reverse inclusion, assume $y \in \cap \{pcl (f [V]) : V \in PO (X), V \supset A\}$…(1) . If possible suppose $f [N_p (x)] \not\ni y$ for any $a \in A$.Then for each $a \in A$, there exist $U (a) \in N_p (a)$ and $V_a \in N_p (y)$ such that $f [U (a)] \cap V_a = \phi$....(2).Now $\{U (a) : a \in A\}$ is a cover of A by p.o. sets in X and precompactness of A provides a finite family$\{U (a_1), U (a_2), \ldots, U (a_n)\}$which covers A. So, $A \subset \cup \{ U (a_i): i= 1,2,\ldots,n\}$...(3). Let $\{V_\alpha : i = 1, 2, \ldots, n\}$ be the corresponding pre-nbds of y satisfying (2). Set $V= \cap \{V_\alpha :i=1,2,\ldots,n\}$.Clearly, $V \in PO (Y, y)$. Also in virtue of (2) $V \cap f [\cup \{ U (a_i) :i=1,2,\ldots,n\}] = \phi \Rightarrow V \cap f [U] = \phi$ where $\cup \{U (a_i): i=1,2,\ldots,n\} = U \in PO (X)$ and $A \subset U$, by (3)$\Rightarrow y \notin \{pcl (f [U]) : U \in PO (X), U \supset A\} \Rightarrow y \notin \cap \{pcl (f [U]) : U \in PO (X), U \supset A\} \Rightarrow$ a contradiction to (1).So, $f [N_p (x)] \ni y$. Consequently, $y \in PCP (f ; a)$ for some $a \in A$ and hence $y \in PCP (f ; A)$.Thus $PCP (f ; A) = \cap \{pcl (f [U]) : U \in PO (X)$ with $U \supset A\}$.So, $PCP (f ; A) \in PC (X)$.

Theorem 3.5. If $G (f) \in PC (X \times Y)$ for the function $f : X \to Y$ and $A \subset X$ is precompact relative to X then $f [A] \in PC (Y)$.

Proof : Since G (f) is preclosed, $PCP (f ; a) = \{f (a)\} \ \forall a \in A = \cup \{PCP (f ; a) : a \in A\} = PCP (f ; A)$.So,$PCP (f ; A) \in PC (Y) \Rightarrow f [A] \in PC (Y)$.

Definition 3.4. Let $f : X \to Y$, $y \in Y$. The inverse pre-cluster set of f at y denoted by $PCP^{-1} (f ; y)$ is the set of all $x \in X$ such that $y \in PCP (f ; x)$.

Theorem 3.6. Let $f : X \to Y$. G (f) is preclosed iff $PCP^{-1} (f ; y) = \{f^{-1} (y)\} \quad \forall y \in Y$.

Proof : Suppose G (f) is preclosed. Let $x \in PCP^{-1} (f ; y)$. Then $y \in PCP (f ; x)$. The preclosedness of G (f), yields that $PCP (f ; x) = \{f (x)\}$. Now $y \in \{f (x)\} \Rightarrow y = f (x) \Rightarrow x \in \{f^{-1} (y)\} \Rightarrow PCP^{-1} (f ; y) \subset \{f^{-1} (y)\}$.To exhibit the reverse inclusion, let $x \in \{f^{-}(y)\}$. So, $f (x) = y$. Since G(f) is preclosed, $PCP (f ; x) = \{f (x)\}$.From this one obtains $y \in PCP (f ; x)$ and hence $x \in PCP^{-1} (f ; y)$ which in its turn gives that $\{f^{-1} (y)\} \subset PCP^{-1} (f ; y)$.So, $PCP^{-1} (f ; y) = \{f^{-}(y)\}$.Conversely let $x \in X$ and $y \in PCP (f ; x)$. By definition, then, $x \in PCP^{-1}(f ; y)$. This, by hypothesis, then gives that $x \in \{f^{-1} (y)\}$. Hence $y = f (x)$. From this it follows that $PCP (f ; x) = \{f (x)\}$,then guarantees the preclosedness of G(f).

Theorem 3.7. Let $f : X \to Y$ and A be precompact relative to Y. Then

$$PCP^{-1} (f ; A) = \cup \{PCP^{-1} (f ; a) : a \in A\} \in PC (X).$$

Proof : Let $x \in PCP^{-1}(f\,;A)$. Then $x \in PCP^{-1}(f\,;a)$, for some $a \in A$. By definition, this implies that $a \in PCP(f\,;x)$.Then $x \in \cap\{pcl(f^{-1}[B]) : B \in N_p(a)\}$.Let $V \in PO(Y)$ with $V \supset A$. This means $V \in N_p(a)$.So, $x \in \cap\{pcl(f^{-1}[V]) : V \in PO(Y) \text{ with } V \supset A\}$, whence $PCP^{-1}(f\,;A) \subset \cap\{pcl(f^{-1}[V] : V \in PO(Y) \text{ with } V \supset A\}$.To prove the reverse inclusion suppose that $x \in \cap\{pcl(f^{-1}[V]) : V \in PO(Y) \text{ with } V \supset A\}$.Suppose, if possible $x \notin PCP^{-1}(f\,;A) \Rightarrow x \notin PCP^{-1}(f\,;a)\ \ \forall a \in A \Rightarrow a \notin PCP(f\,;x)\ \ \forall a \in A$.This indicates that $f^{-1}[N_p(a)] \not\ni x\ \forall a \in A$.Then for each $a \in A$, there exist $V(a) \in PO(Y, a)$ and $U_\alpha \in PO(X, x)$ with $f^{-1}[V_a] \cap U_a = \phi$.Clearly, $\{V(a) : a \in A\}$ is a cover of A by p.o. sets. The precompactness of A provides a finite subfamily $\{V(a_i) : i = 1, 2, \ldots, n\}$ of the above family such that $A \subset \cup V(a_i)$.Let $\{U_a : i = 1, 2, \ldots, n\}$ be the corresponding pre-nbds of x. Now consider the expression $P = Q \cap \{f^{-1}[V(a_1)] \cap f^{-1}[V(a_2)] \cap \ldots . \cap f^{-1}[V(a_n)]\}$where $Q = \cap U_a \in PO(X, x)$, as X enjoys the property P. By the foregoing $P = \phi$. So, $Q \cap f^{-1}[\cup V(a_i)] = \phi \Rightarrow Q \cap f^{-1}[V] = \phi$, where $V = \cup V(a_i) \in PO(Y)$.Now, $V \in PO(Y)$ and $Q \cap f^{-1}[V] = \phi \Rightarrow x \notin \{pcl(f^{-1}[V]) : V \in PO(Y) \text{ with } V \supset A\} \Rightarrow x \notin \cap\{pcl(f^{-1}[V]) : V \in PO(Y) \text{ with } V \supset A\} \Rightarrow$ a contradiction. Hence $x \in PCP^{-1}(f\,;A)$.Thus $PCP^{-1}(f\,;A) = \cap\{pcl(f^{-1}[V]) : V \in PO(Y) \text{ with } V \supset A\}$.This indicates that, $PCP^{-1}(f\,;A) \in PC(X)$.

Theorem 3.8. Let $G(f) \in PC(X \times Y)$ for the function $f : X \to Y$. If A is precompact relative to Y, then $f^{-1}[A] \in PC(X)$.

Proof : Since G(f) is preclosed $PCP^{-1}(f\,;a) = \{f^{-1}(a)\}$ for every $a \in A$.Now $f^{-1}[A] = \cup\{f^{-1}(a) : a \in A\} = \cup\{PCP^{-1}(f\,;a) : a \in A\} = PCP^{-1}(f\,;A)$.But $PCP^{-1}(f\,;A) \in PC(X)$, Hence $f^{-1}[A] \in PC(X)$.

Definition 3.5. A mapping $f : X \to Y$ is called p-closed if $f[F] \in PC(Y)$ for every $F \in PC(X)$.

Lemma 3.1. Let $f : X \to Y$ be a p-closed map. Given any subset S of Y and any $A \in PO(X)$ with $f^{-1}[S] \subset A$, there exists a $B \in PO(Y)$ containing S such that $f^{-1}[B] \subset A$.

Proof : Let $B = Y - f[X - A]$. Since $f^{-1}[S] \subset A$ it follows that $S \subset B$. Moreover $B \in PO(Y)$ as f is p-closed.The fact that $f^{-1}[B] = X - f^{-1}f[X - A] \subset X - (X - A) = A$, completes the proof.

Theorem 3.9. If the function $f : X \to Y$ is p-closed with preclosed point inverses and X is pre-regular then G (f) is preclosed.

Proof : Clearly, we have $\{f^{-1}(y)\} \subset PCP^{-1}(f\,;y)\ \forall y \in Y$. Since $\{f^{-1}(y)\}$ is p.c. for every $y \in Y$, it follows that $pcl(\{f^{-1}(y)\}) = \{f^{-1}(y)\}$. We now assert that $PCP^{-1}(f\,;y) \subset pcl(\{f^{-1}(y)\}) = \{f^{-1}(y)\}\ \ \forall\ y \in Y$.Let $y \in Y$. If possible, there exists a point $x \in X$ such that $x \in PCP^{-1}(f\,;y) - pcl(\{f^{-1}(y)\})$.The pre-regularity of X, then, gives the existence of $U \in PO(X, x)$ and $V \in PO(X)$

containing $\{f^{-1}(y)\}$ such that $U \cap V = \phi \Rightarrow U \cap pcl(V) = \phi$.Since f is p-closed and $\{f^{-1}(y)\} \subset V$,there exists a $W \in PO(Y, y)$ such that $f^{-1}[W] \subset V$.Now $x \in U \Rightarrow x \notin pcl(V) \Rightarrow x \notin pcl(f^{-1}[W]) \Rightarrow x \notin \cap \{pcl(f^{-1}[W]) : W \in N_p(y)\}$.Then $y \notin PCP(f ; x)$, whence $x \notin PCP^{-1}(f ; y)$. But this is a contradiction to the above assumption and hence the preclosedness of G(f) is finally established.

Corollary 3.3. Let $f : X \rightarrow Y$ be p-closed with preclosed point inverses. If X is pre-regular while Y is precompact then f is pc.

Proof : Since f is p-closed with preclosed point inverses and X is pre-regular, $G(f) \in PC(X \times Y)$. The precompactness of Y and preclosedness of G (f) together imply that f is pc.

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