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Wiener and Hyper-Wiener polynomials of Unitary Cayley Graphs

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ABSTRACT:

The two generating functions, namely, Wiener and Hyper-Wiener polynomials are the q-analogues of the topological indices - Wiener and Hyper-Wiener indices respectively. Both polynomials have found substantial applications in chemical graph theory. However, these applications are by no means restricted to molecular graph, but we can also determine a remarkable variety of novel mathematical results. Motivated by this, we computed Wiener and Hyper-Wiener polynomials of Unitary Cayley graphs in this paper.

KEYWORDS: Wiener index, Wiener polynomial, Hyper-Wiener index, Hyper-Wiener polynomial, Unitary Cayley graphs.

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INTRODUCTION:

Throughout this paper, we consider simple connected graph $G = (V, E)$ with n vertices and m edges. We denote the distance between the vertices u and v with $d(u, v)$.

The Wiener polynomial of G, $W(G; q)$, is the polynomial whose first derivative at q =1 gives the Wiener index. i.e., W (G) = W' (G;1). It can be defined as $W(G) = \sum_{\{u,v\}} q^{d(u,v)}$.

Analogously, the Hyper-Wiener polynomial of G, $WW(G; q)$, is the polynomial whose first derivative at q =1 gives the Hyper-Wiener index.

i.e., WW (G) = WW' (G;1). It can be defined as $WW(G) = \sum_{\{u,v\}} q^{d(u,v)+d^2(u,v)}$.

For more detailed study of these polynomials and their respective indices, refer [2-9, 14].

In this paper, we urge to find out the Wiener and Hyper-wiener polynomials of Unitary Cayley graphs. Given a positive integer n > 1, the Unitary Cayley graph, denoted by X_n, can be defined as X_n = Cay(Z_n,U_n), where Z_n is the additive group of ring of integers modulo n and U_n is the multiplicative group of its units. Therefore, its vertex set is Z_n and edge set is $\{(u,v); \gcd(u-v,n)=1\}$, for $u,v \in Z_n$. These graphs have got the property that they have integral spectrum and thus play a vital role in modelling quantum spin network supporting the perfect state transfer. Let ϕ(n) denotes the Euler function. View [1, 10-13, 15] for the comprehensive study of graphs and Unitary Cayley Graphs.

Let us see the following lemma which we use in the theorems:

LEMMA 1.1: [11] *Denote $F_n(s) = F_n(a - b)$, the number of common neighbours of vertices $a \neq b$ in the Unitary Cayley graph X_n for integers a, b, $n \geq 2$ and prime p, . Then $F_n(s)$ is given by*

$$F_n(s) = n \prod_{p/n} \left(1 - \frac{\varepsilon(p)}{p}\right), \text{ where } \varepsilon(p) = \begin{cases} 1, \text{if } p \text{ divides } s \\ 2, \text{if } p \text{ doesnot divide } s \end{cases}$$

WIENER POLYNOMIAL OF UNITARY CAYLEY GRAPHS:

THEOREM 2.1: *If X_n is the Unitary Cayley graph, then the Wiener polynomial of X_n is given by*

$$W(X_n;q)=\begin{cases}\frac{n(n-1)}{2}q,\ if\ n\ is\ prime\\ \frac{n\phi(n)}{2}q+\frac{n(n-2)}{4}q^2,\ if\ n=2^\alpha,\alpha>1\\ \frac{n\phi(n)}{2}q+\frac{n(n-2)}{4}q^2+\frac{n(n-2\phi(n))}{4}q^3,\ if\ n\ is\ even\ and\ has\ an\ odd\ prime\ divisor\\ \frac{n\phi(n)}{2}q+\frac{n(n-\phi(n)-1)}{2}q^2,\ if\ n\ is\ odd\ but\ not\ prime.\end{cases}$$

PROOF: For n is prime, X_n is complete. So $d(u, v) = 1$, $\forall$ u,v $\in X_n$. Therefore, by definition of Wiener polynomial, we obtain $W(X_n;q)=\sum_{\{u,v\}} q^{d(u,v)}=\frac{n(n-1)}{2}q.$

When $n = 2^\alpha$, $\alpha>1$, X_n is complete bipartite with vertex partition $V(X_n)$ = $\{0,2,\ldots,(n-2)\}\cup\{1,3,\ldots,(n-1)\}$. Then it is clear that $d(u, v) = 1$ or 2. As a result, we get a 2-degree polynomial such that $W(X_n;q)=n^2q + n(n-1)q^2$.

Now we take the case of n as even and has an odd prime divisor p, where $n \neq 2^\alpha, \alpha$ >1. This shows that X_n is bipartite with vertex set V as the union of V_1 = $\{0, 2,\ldots, (n-2)\}$ and $V_2 = \{1,3,\ldots,(n-1)\}$. In order to find out the Wiener polynomial of X_n, we need to calculate $d(u, v)$. For the procedure, let us take the condition u $\in V_1$ or u $\in V_2$. First we take u $\in V_1$

Claim 1: $d(u, v)$ =2

Let v $\in V_1$. Clearly, u and v are not adjacent. Then by Lemma 1.1, for u,v $\in V_1$, there exists a common neighbour. So d(u,v) = 2.

Claim 2: $d(u, v)$ = 3

Now, consider the case u $\in V_1$ and v $\in V_2$. It is understood that there exists ϕ(n) neighbours of u in V_2. So we take $V_2 = A\cup B$, where $A=\{v\in V_2;\ uv \in E(X_n)\}$ and $B=\{v\in V_2;\ uv \notin E(X_n)\}$. Obviously, for u u $\in V_1$ and v $\in$ A, d(u,v) = 1.Let v$\in$B. It follows that u and v are not adjacent. So take w $\in A\subset V_2$. Then uw $\in E(X_n)$. But we can see that v and w are both odd. So there should exist a common neighbour x to v and w which results in the conclusion that $d(u, v) = 3$. The case of u $\in V_2$ is analogous to the case u $\in V_1$. Thus it follows by definition of Wiener polynomial,

$$W(X_n;q)=\sum_{\{u,v\}} q^{d(u,v)}=\frac{n\phi(n)}{2}q+\frac{n(n-2)}{4}q^2+\frac{n(n-2\phi(n))}{4}q^3.$$

For n is odd but not prime, assume that $p_1,p_2,\ldots,p_s$ are the different prime divisors of n. Let $n = p_1^{r_1}, p_2^{r_2},\ldots, p_s^{r_s}$, $p_i \neq 2, 1 \leq i \leq s$. Since the factors in the expansion of $F_n(a - b)$ in Lemma1.1 are all postive, all the vertices are either adjacent or there exist a common neighbour to every pair of distinct vertices. This leads to the point that $d(u, v) = 1$ or 2. Hence again using the definition of Wiener polynomial, we reach the result that $W(X_n;q) = \sum_{\{u,v\}} q^{d(u,v)} = \frac{n\phi(n)}{2}q + \frac{n(n-\phi(n)-1)}{2}q^2.$

This completes the proof.

HYPER-WIENER POLYNOMIAL OF UNITARY CAYLEY GRAPHS:

THEOREM 3.1: *If X_n is the Unitary Cayley graph, then the Hyper-Wiener polynomial of X_n is given by*

$$WW(X_n;q) = \begin{cases} \frac{n(n-1)}{2}q^2, \text{ if } n \text{ is prime} \\ \frac{n\phi(n)}{2}q^2 + \frac{n(n-2)}{4}q^6, \text{ if } n = 2^{\alpha}, \alpha > 1 \\ \frac{n\phi(n)}{2}q^2 + \frac{n(n-2)}{4}q^6 + \frac{n(n-2\phi(n))}{4}q^{12}, \text{ if } n \text{ is even and has an odd prime divisor} \\ \frac{n\phi(n)}{2}q^2 + \frac{n(n-\phi(n)-1)}{2}q^6, \text{ if } n \text{ is odd but not prime.} \end{cases}$$

PROOF: The proof is quite direct from the proof of Theorem 2.1.

CONCLUSION:

In this paper, we direct our attention to the two polynomials, namely, Wiener and Hyper-Wiener polynomials. Also, we could form the result with the computation of Wiener and Hyper-Wiener polynomials of Unitary Cayley graphs.

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