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# A Study on Derived Fourier Series Using Banach - Cesaro Sum ability Method

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## ABSTRACT

In this paper, we are presenting the concept of product summability Banach$(C, 1)$ of derived series of the Fourier series. This result is motivated by Dayal[1].

**KEYWORDS:** $(C, 1)$ summability; Banach summability; Banach$(C, 1)$ summability.

**MATHEMATICS SUBJECT CLASSIFICATION (2010):** 42A24, 42A20 and 42B08

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## INTRODUCTION

The concept of sum ability theory has arises so speedly in recent years. It came after the summation of non-convergent series and has played an important role in the field of quantum mechanics, Fourier analysis, Fixed point theory, probability theory and so many.

S. Banach[2] extends the limit function defined on the space of all convergent sequences to the space of all bounded sequences. Furthermore, Lorentz[3] has derived the concept of almost convergent through the Banach limit in the year 1948.

Lorentz[3] gave an idea for summation by assigning a general limit $x_n$ for the bounded sequences $x = \{x_n\}$. In the theory of almost periodic function this method is similar to the mean values. It can be seen that this general limit is narrowly connected to that of limits of S. Banach[2].

According to Mishra[4], now we are defining Banach summability

Let Ω and $l_\infty$ denote the linear space of all sequences and bounded sequences of real numbers respectively. A linear functional L on $l_\infty$ is said to be Banach limit if and only if the function L satisfies the following properties

(1.1) $L_1$ : $L(x) \geq 0$

for every $x \geq 0$ i.e. $x_n \geq 0, \forall n \in N, x \in l_\infty$

(1.2) $L_2$ : $L(e) = 1$

for $e = (1,1,1, \ldots \ldots \ldots .)$

(1.3) $L_3$ : $L(x) = L(\tau(x))$

for every $x = \{x_n\} \in l_\infty$

where, τ denotes the shift operator on $l_\infty$ such that

$\tau(x) = \tau(\{x_n\}) = \{x_{n+1}\}$

A sequence $x \in l_\infty$ is a Banach summable if all the Banach limits of x are the same.

If L is a limit functional on $l_\infty$ , then

$\forall\, x \in l_\infty$ ,

we say that, $L(x)$ is called a Banach limit of x.

A series $\sum u_n$ with sequence of its partial sums $\{s_n\}$ is said to be Banach summable iff $\{s_n\}$ is Banach summable.

Let $t_n$ be the sequence defined by

$$t_n = \frac{1}{n+1}\sum_{k=p}^{n+p} s_k\,, \quad n \in N,$$

then $t_n$ is said to be the $k$-th element of the Banach transformed sequence. If

$$\lim_{n\to\infty} t_n = s\,,$$

a finite number , uniformly $\forall\, n \in N$ , then

$\sum u_n$ is said to be Banach summable to $s$.

The $(C,1)$ transform is defined as the $n^{th}$ partial sum of $(C,1)$ summability (Cesaro summability) Titmarsh[5] and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s$$

as $n \to \infty$ , then

the infinite series $\sum u_n$ is summable to the definite number $s$ by $(C,1)$ method.

If the method of Banach summability is superimposed on the Cesaro means of order one, another method of summability Banach $(C,1)$ is obtained.

Then the series $\sum u_n$ is said to be summable by Banach $(C,1)$ means or summable Banach $(C,1)$ to a definite number $s$ i.e.

$$t_k(p) = \frac{1}{k+1}\sum_{\upsilon=p}^{p+k}\left\{\frac{1}{n+1}\sum_{k=0}^{n} S_k\right\} \to s$$

as $n \to \infty$ .

We shall consider a function $f(x)$ of bounded variation integrable in the sense of Lebesgue and periodic with period$2\pi$.

If,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

then $f(t)$ generates the Fourier –Lebesgue series

$$(1.4) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt) \equiv \sum_{n=0}^{\infty} A_n(t)$$

The series

$$(1.5) \quad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=0}^{\infty} nB_n(t)$$

which is obtained by differentiating (1.4) term by term is called the derived Fourier series or the derived series of $f(t)$.

We write,

$$\emptyset(t) = f(x+t) + f(x-t) - 2s$$

$$g(t) = \frac{\Psi(t)}{4\sin\frac{t}{2}}$$

$$\Psi(t) = f(x+t) - f(x-t)$$

## KNOWN RESULTS

Dayal[1] has introduced the concept of Banach summability first time in the field of Fourier series showing the existence of unique Banach limit of the series given yields a concept for the convergence of Fourier series. He proved

***Theorem A.*** If

$$(2.1) \qquad \int_0^t |\emptyset_x(u)|\, du = o(t), \quad \text{as } t \to 0+$$

and

$$(2.2) \qquad \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} \frac{|\emptyset_x(u)|}{u} du = 0, \quad \text{as } n \to \infty$$

uniformly with respect to $p$.

Then the Fourier series $f(t)$ has a unique Banach limit and the limit is zero.

Further, Diwan[6] has obtained an analogous result by generalised theorem A for the derived Fourier series. She proved

***Theorem B.*** If

$$(2.3) \qquad G(t) \equiv \int_0^t |g(u)|\, du = o(t), \quad \text{as } t \to 0+$$

and

$$(2.4) \qquad \lim_{n\to\infty} \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} \frac{|g(u)|}{u} du = o(1)$$

hold uniformly with respect to $p$, then the derived Fourier series $(1.5)$ has a unique Banach limit and this limit is zero .

Dealing with Banach summability, many researchers gave an idea for $F_A$- limit and Banach limit of Fourier series like Ogra[7], Kori[8] and Aizpura et al.[9] and so on.

## MAIN RESULT

In this paper, our object is to study the concept of product summability of Banach summable over Cesaro mean by generalizing the previous results. We have given a new result for the Banach Cesaro summablility of derived Fourier series by taking the same conditions. The Theorem is as follows:

***Theorem C***. If

$$(3.1) \qquad G(t) = \int_0^t |g(u)|\, du = o(t),$$

as $t \to 0$

and

$$(3.2) \qquad \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} \frac{|g(u)|}{u} du = 0,$$

as $n \to \infty$, uniformly w.r.to 'p'

then the derived Fourier series $(1.5)$ is summable Banach $(C, 1)$ to zero at the point x.

We require the lemma for our proof.

## LEMMA

If we write

$$K_n^p(t) = \frac{-1}{(n+1)} \sum_{\upsilon=p}^{n+p} \frac{\sin(\upsilon+1)t}{\sin\frac{t}{2}}$$

then

$$K_n^p(t) = \begin{cases} O(n+p+1), & \text{for } 0 \le t \le \frac{1}{n+p+1} \\ O\left(\frac{1}{t}\right), & \text{for } 0 \le t \le \pi \\ O\left(\frac{1}{nt^2}\right), & \text{for } \frac{1}{n+1} \le t \le \pi \end{cases}$$

***Proof of lemma***

for $0 \le t \le \frac{1}{n+p+1}$,

we have,

$$\left|K_n^p(t)\right| = \left|\frac{1}{(n+1)}\sum_{\upsilon=p}^{n+p} O(\upsilon+1)\right|$$

(4.1) $= O(n+p+1)$

$$\because \frac{\sin(\upsilon+1)t}{\sin\frac{t}{2}} = O(\upsilon+1)$$

Also, for $0 \leq t \leq \pi$ ,

We have, on simplification

$$\left|K_n^p(t)\right| = \left|\frac{\sin\left(p+\frac{n}{2}+1\right)t\sin(n+1)\frac{t}{2}}{(n+1)\left(\sin\frac{t}{2}\right)^2}\right|$$

(4.2) $= O\left(\frac{1}{t}\right)$

Similarly,

for $\frac{1}{n+1} \leq t \leq \pi$

$$\left|K_n^p(t)\right| = \left|\frac{\cos pt - \cos(n+p+1)t + \cos(p+1)t - \cos(n+p+2)t}{2(n+1)\sin t.\sin\frac{t}{2}}\right|$$

$$\leq \left|\frac{\cos pt - \cos(n+p+1)t}{2(n+1)\sin t.\sin\frac{t}{2}}\right| + \left|\frac{\cos(p+1)t - \cos(n+p+2)t}{2(n+1)\sin t.\sin\frac{t}{2}}\right|$$

$= I_{1.1} + I_{1.2}$ , say

$$\therefore \quad I_{1.1} = \left|\frac{\cos pt - \cos(n+p+1)t}{2(n+1)\sin t.\sin\frac{t}{2}}\right| = O\left(\frac{1}{nt^2}\right)$$

By virtue of [6].

Also,

$$I_{1.2} = \left|\frac{\cos(p+1)t - \cos(n+p+2)t}{2(n+1)\sin t.\sin\frac{t}{2}}\right|$$

(4.3) $= O\left(\frac{1}{nt^2}\right)$

## PROOF OF THEOREM C

The $n^{th}$ partial sum $S'_n$ of derived Fourier series (1.5) given by Titmarch[5] can be written as

$$S'_n = \frac{2}{\pi}\int_0^{\pi}\Psi(t)\left[\sum_{r=1}^{n} r\sin rt\right]dt$$

$$= \frac{-2}{\pi}\int_0^{\pi}\Psi(t)\frac{d}{dt}\left[\frac{1}{2}+\sum_{r=1}^{n}\cos rt\right]dt$$

$$= \frac{-1}{\pi}\int_0^{\pi}\Psi(t)\frac{d}{dt}\left[\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right]dt$$

$(C, 1)$ transform of $S'_n$ will be given by

$$C'_n = \frac{-1}{(n+1)\pi}\int_0^{\pi}\Psi(t)\frac{d}{dt}\left[\frac{1-\cos(n+1)t}{\sin^2\frac{t}{2}}\right]dt$$

$$= \frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\cos\frac{t}{2}\{1-\cos(n+1)t\}}{(n+1)\sin^2\frac{t}{2}} - \frac{\sin(n+1)t}{\sin\frac{t}{2}}\right]dt$$

$$C'_n = \frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\cos\frac{t}{2}\{1-\cos(n+1)t\}}{(n+1)\sin^2\frac{t}{2}}\right]dt$$

$$- \frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\sin(n+1)t}{\sin\frac{t}{2}}\right]dt$$

$(5.1)$ $\sim I_1 + I_2$ , say

Let,

$$I_1 = \frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\cos\frac{t}{2}\{1-\cos(n+1)t\}}{(n+1)\sin^2\frac{t}{2}}\right]dt$$

$$= \frac{1}{\pi}\int_0^{\delta} g(t)\left[\frac{2\{1-\cos(n+1)t\}}{(n+1)t^2}\right]dt + o(1),$$

as $n \to \infty$

since the last interval is $o(1)$, by the presence of $n$ in denominator & by Riemann-Lebesgue Theorem.

If $(1.10)$ hold for $t \le \delta$ , then

$$\frac{2}{\pi}\int_0^{\delta} g(t)\left[\frac{2\{1-\cos(n+1)t\}}{(n+1)t^2}\right]dt = o(1),$$

due to Saxena[10]

$\therefore\ I_1 = o(1)$

It follows from $(5.1)$ that

$$C'_n = -\frac{1}{\pi}\int_0^{\pi} g(t)\left[\frac{\sin(n+1)t}{\sin\frac{t}{2}}\right]dt$$

Banach $(C, 1)$ transform of $S_n$ is given by,

$$\sigma'_n = \frac{-1}{(n+1)\pi}\int_0^{\pi} g(t)\sum_{\upsilon=p}^{n+p}\frac{\sin(\upsilon+1)t}{\sin\frac{t}{2}}dt + o(1)$$

$$= \frac{1}{\pi}\int_0^{\pi} g(t)K_n^p(t)dt + o(1)$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{1}{n+p+1}} + \int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} g(t)K_n^p(t)dt\right] + o(1)$$

(5.2) $= \frac{1}{\pi}[P + Q + R] + o(1)$, say

By using condition (3.1) and (4.1), we have

$$|P| = \frac{1}{\pi}O(n+p+1)\int_0^{\frac{1}{n+p+1}}|g(t)|dt$$

(5.3) $= o(1)$, as $n \to \infty$

On applying condition (3.2) and (4.2),

we have

$$|Q| = \frac{1}{\pi}\int_{\frac{1}{n+p+1}}^{\frac{1}{n+1}} O\left(\frac{1}{t}\right)|g(t)|dt$$

(5.4) $= o(1)$, as $n \to \infty$

Also,

By using (3.1) and integration by part, we have

$$|R| = O\left(\frac{1}{n}\right)\frac{1}{\pi}\int_{\frac{1}{n+1}}^{\pi} O\left(\frac{1}{t^2}\right)|g(t)|dt$$

$$= O\left(\frac{1}{n}\right)\frac{1}{\pi}\left[\left\{\frac{G(t)}{t^2}\right\}_{\frac{1}{n+1}}^{\pi} + 2\int_{\frac{1}{n+1}}^{\pi}\frac{G(t)}{t^3}dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[O\left(\frac{1}{t}\right)\right]_{\frac{1}{n+1}}^{\pi}$$

(5.5) $= o(1)$,

as $n \to \infty$

By virtue of (5.2), (5.3), (5.4) and (5.5),

$\sigma'_n = o(1)$, as $n \to \infty$

uniformly w. r. to $p$.

Hence, the proof of Theorem C is complete.

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