Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

Transient solution of a m/m/4 queue with heterogeneous servers subject to catastrophes and balking

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ABSTRACT

This paper demonstrates a transient solution for the system size in a M/M/4 queue where the service rates of the servers are not similar with the possibility of catastrophes at the system also with the impatient customer behavior. For the customer in the system the time dependent probabilities are derived.

KEYWORDS: Transient Analysis, System Size, Heterogeneous Servers, Catastrophes, and Balking.

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1. INTRODUCTION

In the study of queue networks one typically tries to obtain the equilibrium distribution of the network, although in many applications the study of the transient state is fundamental. The transient response is necessarily tied to any event that affects the equilibrium of the system.

Multi-server queuing systems arrive in congestion problems of telephone exchange and computer networks. A complete description of situations with such queuing analysis of computer systems can be found in Lavenberg [14]. In many real multi-server queuing situations, the service with heterogeneity is a common feature. The heterogeneous servers to the waiting lines are analyzed by Gumbel.H [3]. The role of quality and service performance is crucial aspects in customer perceptions and firms must dedicate special attention to them with designing and implementing their operations. For these reasons, the queues with heterogeneity have received considerable attention in the literature. Transient solution of a two processor heterogeneous system has been discussed by Dharamaraja. S [6]. A control model for a machine center with two heterogeneous system has been introduced by Liu and Kumar [7]. A treaties on the Theory of Bessel functions where discussed by Watson. G. N [8]. Whitt. W [9] has analyzed the Untold Horrors of the Waiting Room: What the Equilibrium Distribution Will Never Tell about the Queue Length Process. A research on Measures for Time Dependent Queueing Problem with Service in Batches of Variables Size was done by Garg. P. C [10].

In recent times, queuing model with catastrophes has been investigated by Boucherie and Boxma [11], Jain and Sigman [13] and Dudin and Nishimura [12]. Transient solution of a single server queue with catastrophes are discussed by Kumar,B.K and Arivudainambi.D[4]. An analysis made on the queuing network model with catastrophes and its product from solution by Chao.X [5]. The catastrophes may come either from outside of the system or from another service station of the system. Height[15] first presented the single server queue with balking. Al-seedy and kotb[16] considered the transient solution of a single-server system with balking concept. Al-seedy and et.al[17], studied about transient solution of the c server queue with balking and reneging.

A combined analysis of queues with heterogeneous servers subject to catastrophes to find transient solution of an M/M/2 model by Kumar.B.K, Pavai.M and Vankatakrihnan [1]. Transient solution of a Markovian queuing model with heterogeneous servers and catastrophes has been discussed by Dharmaraja and Rakesh Kumar [2]. Julia Rose Mary and Maria Remona[18] studied the Transient Solution to the M/M/4 heterogeneous servers queueing system subject to catastrophes.

From the output of this study, the queueing system is organized as follows: To describe the queueing model of four server heterogeneous system with balking also with catastrophes and to derive the time-dependent state probabilities for the system size.

2. SYSTEM MODEL

By examining on M/M/4 queuing system with heterogeneous servers and assume that the servers times follow exponential distributions with the service rates as $\mu_1, \mu_2, \mu_3, and\ \mu_4$ for four different servers where $\mu_1 > \mu_2 > \mu_3 > \mu_4$. Consider the customer arrival process in Poisson with rate λ and system also has one waiting line. FCFS queuing discipline is followed and each customer requires exactly one server for the service. When the server becomes free, the customer who is first in the waiting line will join the queue.

Other than arrival and service processes, there also occur catastrophes at the service facilities with rate η in a Poisson manner. In the system, whenever a catastrophe occurs it destroys all the customers in the system immediately, and also the server get inactivated. Then the service is started when a new arrival occurs.

Let $\{X(t), t \in \Re^+\}$ be the number of customers in time *t*. Let $P_n(t) = P(X(t) = n)$, $n = 4,5,6,\ldots$ denotes the probability of n customers in the system at time *t*. Also let $P_0(t) = P(X(t) = 0)$ be the probability that the system is empty at time *t*, $P_1(t) = P(X(t) = 1)$ be the probability that there is one customer in the system, $P_2(t) = P(X(t) = 2)$ be the probability that there are two customers in the system, and $P_3(t) = P(X(t) = 3)$ be the probability that there are three customers in the system.

In this system, we deal with the M/M/4 queueing system with heterogeneous servers subject to catastrophe as well as balking. We consider the customers arrive at the system one by one according to a Poisson process with rate λ. On arrival a customer either decides to join the queue with probability

$\rho = prob\{a\ unit\ joins the\ queue\}$ or balk with probability $1-\rho$, where $0 \le \rho < 1$ if $n = c(1)\infty$ and $\rho = 1$ if $n = 0(1)\overline{c-1}$ where $c = 4$.

3. THE TRANSIENT PROBABILITIES FOR THE QUEUEING SYSTEM

From the above assumptions the transient-state probabilities of heterogeneous servers $P_0(t), P_1(t), P_2(t), P_3(t)\ and\ P_n(t),\quad n = 4,5,6\ldots$ satisfy the following system of differential difference equations with balking and catastrophes:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu_1 P_1(t) + \eta(1 - P_0(t)) \tag{3.1}$$

$$\frac{dP_1(t)}{dt} = -(\lambda + \mu_1 + \eta)P_1(t) + \lambda P_0(t) + (\mu_1 + \mu_2)P_2(t) \tag{3.2}$$

$$\frac{dP_2(t)}{dt} = -(\lambda + \mu_1 + \mu_2 + \eta)P_2(t) + \lambda P_1(t) + (\mu_1 + \mu_2 + \mu_3)P_3(t) \tag{3.3}$$

$$\frac{dP_3(t)}{dt} = -(\lambda + \mu_1 + \mu_2 + \mu_3 + \eta)P_3(t) + \lambda P_2(t) + \mu P_4(t) \tag{3.4}$$

$$\frac{dP_4(t)}{dt} = -(\rho\lambda + \mu + \eta)P_4(t) + \lambda P_3(t) + \mu P_5(t) \tag{3.5}$$

$$\frac{dP_n(t)}{dt} = -(\rho\lambda + \mu + \eta)P_n(t) + \rho\lambda P_{n-1}(t) + \mu P_{n+1}(t),\ n = 5,6,\dots \tag{3.6}$$

where $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$.

Suppose at time t=0 there is no customer in the system, so that $P_0(t)=1$. By using a probability generating function technique the above system of equations are solved. By letting,

$$P(z,t) = G_0(t) + \sum_{n=0}^{\infty} P_{n+5}(t) z^{n+1} \tag{3.7}$$

where $G_0(t) = P_0(t) + P_1(t) + P_2(t) + P_3(t) + P_4(t)$, with initial condition $P(z,0)=1$.

Apply the standard generating function argument, the system of equations 3.1 to 3.6 then yields

$$\frac{\partial P(z,t)}{\partial t} = \eta(1 - G_0(t)) + \lambda(z-1)P_4(t) + \left[\rho\lambda z + \frac{\mu}{z} - (\rho\lambda + \mu + \eta)\right][P(z,t) - G_0(t)] \tag{3.8}$$

Examine equation 3.8 as a first order linear differential equation in $P(z,t)$ and solving, we get ,

$$P(z,t) = e^{Bt} + \int_0^t \{\eta(1 - G_0(u)) + \lambda(z-1)P_4(u) - BG_0(u)\} e^{B(t-u)} du \tag{3.9}$$

where $B = \left[\rho\lambda z + \frac{\mu}{z} - (\rho\lambda + \mu + \eta)\right]$

By utilizing the Bessel function generating function, if $\alpha = 2\sqrt{\rho\lambda\mu}$ *and* $\beta = \sqrt{\frac{\rho\lambda}{\mu}}$, then

$$e^{\left(\rho\lambda z + \frac{\mu}{z}\right)t} = \sum_{n=-\infty}^{\infty} I_n(\alpha t)(\beta z)^n$$

where $I_n(.)$ is the modified Bessel function of first kind of order n.

Equating this in equation 3.9, then expanding $P(z,t)$ as a series in z and comparing the co-efficient of z^n on either side, we get for $n = 1,2,3,\dots$

$$P_{n+4}(t) = \beta^n I_n(\alpha t) e^{-bt} + \eta\beta^n \int_0^t (1 - G_0(u)) I_n(\alpha(t-u)) e^{-b(t-u)} du$$

$$+ \lambda\beta^n \int_0^t P_4(u)\left[I_{n-1}(\alpha(t-u))\beta^{-1} - I_n(\alpha(t-u))\right] e^{-b(t-u)} du$$

$$-\beta^{n}\int_{0}^{t}\left[\rho\lambda I_{n-1}(\alpha(t-u))\beta^{-1}+\mu\beta I_{n+1}(\alpha(t-u))-bI_{n}(\alpha(t-u))\right]G_{0}(u)e^{-b(t-u)}du \tag{3.10}$$

where $b=\rho\lambda+\mu+\eta$ and further, when n=0, we get

$$\begin{aligned}\beta G_{0}(t)&=\beta I_{0}(\alpha t)e^{-bt}+\eta\beta^{n}\int_{0}^{t}(1-G_{0}(u))I_{0}(\alpha(t-u))e^{-b(t-u)}du\\&+\lambda\int_{0}^{t}\mathrm{P}_{4}(u)[I_{1}(\alpha(t-u))-\beta I_{0}(\alpha(t-u))]e^{-b(t-u)}du\\&-\int_{0}^{t}\left[2\rho\lambda I_{1}(\alpha(t-u))\beta^{-1}-b\beta I_{0}(\alpha(t-u))\right]G_{0}(u)e^{-b(t-u)}du\end{aligned} \tag{3.11}$$

where we have used $I_{-n}(.)=I_{n}(.)$.

Since $\mathrm{P}(z,t)$ does not contain terms with negative powers of z, the right hand side of 3.10 with n replaced with –n must be zero. Thus,

$$\begin{aligned}0&=\beta^{n}I_{n}(\alpha t)e^{-bt}+\eta\beta^{n}\int_{0}^{t}(1-G_{0}(u))I_{n}(\alpha(t-u))e^{-b(t-u)}du\\&+\lambda\beta^{n}\int_{0}^{t}\mathrm{P}_{4}(u)[I_{n+1}(\alpha(t-u))\beta^{-1}-I_{n}(\alpha(t-u))]e^{-b(t-u)}du\\&-\beta^{n}\int_{0}^{t}\left[\rho\lambda I_{n+1}(\alpha(t-u))\beta^{-1}+\mu\beta I_{n-1}(\alpha(t-u))-bI_{n}(\alpha(t-u))\right]G_{0}(u)e^{-b(t-u)}du\end{aligned} \tag{3.12}$$

Utilizing equation 3.12 in 3.10, after some algebraic manipulation, we obtain for n=1,2,3,…

$$\mathrm{P}_{n+4}(t)=n\beta^{n}\int_{0}^{t}\mathrm{P}_{4}(u)\frac{I_{n}(\alpha(t-u))}{(t-u)}e^{-b(t-u)}du \tag{3.13}$$

4. THE TRANSIENT PROBABILITY $\mathrm{P}_4(t)$:

So far, the probabilities $\mathrm{P}_0(t),\mathrm{P}_1(t),\mathrm{P}_2(t),\mathrm{P}_3(t)$ *and* $\mathrm{P}_4(t)$ remain to be found. To find, we consider the system of equations 3.1 to 2.4 subject to condition 3.11. Equations 3.1 to 3.4 can be expressed in matrix from as

$$\frac{d\mathrm{P}(t)}{dt}=\mathrm{M}\mathrm{P}(t)+\eta e_{1}+\mu\mathrm{P}_{4}(t)e_{2} \tag{4.1}$$

where $\mathrm{P}(t)=(\mathrm{P}_0(t),\mathrm{P}_1(t),\mathrm{P}_2(t),\mathrm{P}_3(t))^{\mathrm{T}}$, $e_1=(1,0,0,0)^{\mathrm{T}}$ and $e_2=(0,0,0,1)^{\mathrm{T}}$,

$$\mathrm{M}=\begin{pmatrix}-\lambda+\eta & \mu_1 & 0 & 0\\ \lambda & -(\lambda+\mu_1+\eta) & \mu_1+\mu_2 & 0\\ 0 & \lambda & -(\lambda+\mu_1+\mu_2+\eta) & \mu_1+\mu_2+\mu_3\\ 0 & 0 & \lambda & -(\lambda+\mu_1+\mu_2+\mu_3+\eta)\end{pmatrix}$$

In continuation, let $P_n^*(s)$ denote the Laplace transform of $P_n(t)$. Now, by taking Laplace transforms, the result of 4.1 is obtained as

$$P^*(s)=(sI-M)^{-1}\left[\left(1+\frac{\eta}{s}\right)e_1+\mu P_4^*(s)e_2\right] \tag{4.2}$$

with $P(0)=(1,0,0,0)^T$ (4.3)

Hence, only $P_4^*(s)$ is to be found. We note that, if $e=(1,1,1,1)^T$,

$$G_0^*(s)=e^T P^*(s)+P_4^*(s) \tag{4.4}$$

Taking Laplace transforms, after simplification, equation 2.11 yields,

$$G_0^*(s)=\frac{1}{s}+\frac{1}{2(s+\eta)}P_4^*(s)\left[\Omega-\sqrt{\Omega^2-\alpha^2}-2\rho\lambda\right] \tag{4.5}$$

where $\Omega=s+\rho\lambda+\mu+\eta$

Utilizing 4.5 in 4.4 and solving for $P_4^*(s)$, we obtain

$$P_4^*(s)=\frac{\left(1+\frac{\eta}{s}\right)\left[1-e^T(sI-M)^{-1}(s+\eta)e_1\right]}{(s+\eta+\rho\lambda)-\frac{1}{2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]+e^T(sI-M)^{-1}(s+\eta)\mu e_2} \tag{4.6}$$

Let $(sI-M)^{-1}=\left(m_{ij}^*(s)\right)_{4\times 4}$

It is easy to see that,

$$(sI-M)^{-1}=\frac{\begin{pmatrix} g_1(s)i(s)-\lambda g_3(s)(\mu_1+\mu_2) & -\mu_1 i(s) & g_3(s)\mu_1(\mu_1+\mu_2) & -\mu_1(\mu_1+\mu_2)(\mu-\mu_4) \\ -\lambda i(s) & f(s)i(s) & -f(s)g_3(s)(\mu_1+\mu_2) & f(s)(\mu_1+\mu_2)(\mu-\mu_4) \\ \lambda^2 g_3(s) & -\lambda f(s)g_3(s) & g_3(s)(f(s)g_1(s)-\lambda\mu_1) & (\lambda\mu_1-f(s)g_1(s))(\mu-\mu_4) \\ -\lambda^3 & \lambda^2 f(s) & \lambda(\lambda\mu_1-f(s)g_1(s)) & f(s)j(s)-\mu_1\lambda g_2(s) \end{pmatrix}}{|D(M)|} \tag{4.7}$$

where $g_1(s)=s+\lambda+\mu_1+\eta$; $g_2(s)=s+\lambda+\mu_1+\mu_2+\eta$; $g_3(s)=s+\lambda+\mu_1+\mu_2+\mu_3+\eta$;
$f(s)=s+\lambda+\eta$; $i(s)=g_2(s)g_3(s)-\lambda(\mu-\mu_4)$; $j(s)=g_1(s)g_2(s)-\lambda(\mu_1+\mu_2)$.

and

$$|D(M)|=s^4+s^3(4(\lambda+\eta)+3\mu_1+2\mu_2+\mu_3)+s^2[(\lambda+\mu_1+\eta)2(3(\lambda+\eta)+2\mu_2+\mu_3)+(\mu_2+\eta)(\mu_2+\mu_3)$$
$$+3\mu_1(\mu_1+\eta)+\mu_2\eta]+\quad s\{(\lambda+\eta)[(\lambda+\eta)(3(\lambda+\eta)+2\mu_2+\mu_3)+(\mu_2+\mu_3)(\mu_2+\eta)+\mu_2\eta]$$
$$+\lambda\left[\lambda^2+(\lambda+\eta)(3\mu_1+3\eta)+(\eta+\mu_1)(2\mu_1+2\mu_2+\mu_3)+7\mu_1\eta\right]+(\mu_1+\mu_2)(\mu_1\mu_2+\mu_1\mu_3)$$

$$+\eta\left[\eta^2+(\eta+4\mu_1)(\mu_1+2\mu_2+\mu_3)+\mu_1(2\mu_1+8\eta)+\mu_2(\mu_2+\mu_3)\right]+\mu_1^{\ 2}(\mu_1+\mu_2)\ \}+$$
$$\{\ (\lambda+\eta)^2\left[2\mu_2\eta+\mu_3\eta+\lambda^2+\eta^2+2\lambda\eta\right]+(\lambda+\eta)\left[2\mu_1^{\ 2}\eta+3\mu_1\mu_2\eta+\mu_1\mu_3\eta+3\lambda\mu_1\eta\right]$$
$$+\left(\mu_1^{\ 2}\eta+\mu_1\mu_2\eta\right)(\mu_1+\mu_2+\mu_3+\eta)+\mu_1\mu_3\eta^2\ \}$$

The characteristics roots of the matrix M are given by

$$|D(\mathrm{M})|=0 \tag{4.8}$$

By defining,

$$a=\frac{1}{16}\{4[(\lambda+\mu_1+\eta)2(3(\lambda+\eta)+2\mu_2+\mu_3)+(\mu_2+\eta)(\mu_2+\mu_3)+3\mu_1(\mu_1+\eta)+\mu_2\eta]$$
$$-[4(\lambda+\eta)+3\mu_1+2\mu_2+\mu_3]^2\}$$

$$b=\frac{1}{64}\{2(4(\lambda+\eta)+3\mu_1+2\mu_2+\mu_3)^3-16[(\lambda+\mu_1+\eta)2(3(\lambda+\eta)+2\mu_2+\mu_3)+(\mu_2+\eta)(\mu_2+\mu_3)$$
$$+3\mu_1(\mu_1+\eta)+\mu_2\eta][(\lambda+\eta)[(\lambda+\eta)(3(\lambda+\eta)+2\mu_2+\mu_3)+(\mu_2+\mu_3)(\mu_2+\eta)+\mu_2\eta]$$
$$+\lambda\left[\lambda^2+(\lambda+\eta)(3\mu_1+3\eta)+(\eta+\mu_1)(2\mu_1+2\mu_2+\mu_3)+7\mu_1\eta\right]+(\mu_1+\mu_2)(\mu_1\mu_2+\mu_1\mu_3)$$
$$+\eta\left[\eta^2+(\eta+4\mu_1)(\mu_1+2\mu_2+\mu_3)+\mu_1(2\mu_1+8\eta)+\mu_2(\mu_2+\mu_3)\right]+\mu_1^{\ 2}(\mu_1+\mu_2)\Big]+$$
$$64\left[(\lambda+\eta)^2\left[2\mu_2\eta+\mu_3\eta+\lambda^2+\eta^2+2\lambda\eta\right]+(\lambda+\eta)\left[2\mu_1^{\ 2}\eta+3\mu_1\mu_2\eta+\mu_1\mu_3\eta+3\lambda\mu_1\eta\right]\right.$$
$$\left.+\left(\mu_1^{\ 2}\eta+\mu_1\mu_2\eta\right)(\mu_1+\mu_2+\mu_3+\eta)+\mu_1\mu_3\eta^2\right]\}$$

$n=2\sqrt{-a}$ *and* $\theta=\frac{1}{3}\cos^{-1}\left\{-\frac{b}{2\sqrt{-a^3}}\right\}$, the characteristic roots of 4.8 are

$$s_i=n\cos\left[\theta+(i-2)\frac{2\pi}{3}\right]-\frac{(4(\lambda+\eta)+3\mu_1+2\mu_2+\mu_3)}{3},\quad i=1,2,3,4. \tag{4.9}$$

It is examined that $m_{kj}^{\ *}(s)$ are all rational algebraic functions of s. Then, the inverse transform $m_{kj}(t)$ of $m_{kj}^{\ *}(s)$ is obtained by partial fraction decompositions. Since the characteristics roots s_i, $i=1,2,3,4$ of M are all real and distinct, $m_{kj}(t)$ is the inverse transform of $m_{kj}^{\ *}(s)$, which are given by,

$$m_{11}(t)=\sum_{k=1}^{4}\frac{g_1(s_k)[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)]-\lambda g_3(s_k)(\mu_1+\mu_2)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{13}(t)=\sum_{k=1}^{4}\frac{g_3(s_k)\mu_1(\mu_1+\mu_2)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{12}(t)=\sum_{k=1}^{4}\frac{-\mu_1[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)]}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{14}(t)=\sum_{k=1}^{4}\frac{-\mu_1(\mu_1+\mu_2)(\mu-\mu_4)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{21}(t)=\sum_{k=1}^{4}\frac{-\lambda[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)]}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{22}(t)=\sum_{k=1}^{4}\frac{f(s_k)[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)]}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{23}(t)=\sum_{k=1}^{4}\frac{-f(s_k)g_3(s_k)(\mu_1+\mu_2)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{24}(t)=\sum_{k=1}^{4}\frac{f(s_k)(\mu_1+\mu_2)(\mu-\mu_4)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{31}(t)=\sum_{k=1}^{4}\frac{\lambda^2 g_3(s_k)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{32}(t)=\sum_{k=1}^{4}\frac{-\lambda f(s_k)g_3(s_k)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{33}(t)=\sum_{k=1}^{4}\frac{g_3(s_k)(f(s_k)g_1(s_k)-\lambda\mu_1)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{34}(t)=\sum_{k=1}^{4}\frac{(\lambda\mu_1-f(s_k)g_1(s_k))(\mu-\mu_4)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{41}(t)=\sum_{k=1}^{4}\frac{-\lambda^3}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{42}(t)=\sum_{k=1}^{4}\frac{\lambda^2 f(s_k)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

$$m_{43}(t)=\sum_{k=1}^{4}\frac{\lambda(\lambda\mu_1-f(s_k)g_1(s_k))}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

and

$$m_{44}(t)=\sum_{k=1}^{4}\frac{f(s_k)[g_1(s_k)g_2(s_k)-\lambda(\mu_1+\mu_2)]-\mu_1\lambda g_2(s_k)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)}e^{s_k t}$$

From the matrix 4.7, we achieve,

$$e^{\mathrm{T}}(sI-\mathrm{M})^{-1}(s+\eta)e_1=(s+\eta)\sum_{j=1}^{4}m_{j1}{}^{*}(s) \tag{4.10}$$

and

$$e^{\mathrm{T}}(sI-\mathrm{M})^{-1}(s+\eta)\mu e_2=(s+\eta)\mu\sum_{j=1}^{4}m_{j4}{}^{*}(s) \tag{4.11}$$

Replacing 4.10 and 4.11 in 4.6, we get

$$\mathrm{P}_4{}^{*}(s)=\frac{\left(1+\frac{\eta}{s}\right)\left[1-(s+\eta)\sum_{j=1}^{4}m_{j1}{}^{*}(s)\right]}{(s+\eta+\lambda)-\frac{1}{2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]+\mu(s+\eta)\sum_{j=1}^{4}m_{j4}{}^{*}(s)} \tag{4.12}$$

Utilizing equation 4.7 in 4.2, we have

$$\mathrm{P}_0{}^{*}(s)=\frac{1}{|D(\mathrm{M})|}\left\{\left(1+\frac{\eta}{s}\right)(g_1(s)i(s)-\lambda g_3(s)(\mu_1+\mu_2))+\mathrm{P}_4{}^{*}(s)\mu[-\mu_1(\mu_1+\mu_2)(\mu-\mu_4)]\right\}$$

$$=\left(1+\frac{\eta}{s}\right)m_{11}^{*}(s)+\mu\, m_{14}^{*}(s)\mathrm{P}_4^{*}(s) \tag{4.13}$$

$$\mathrm{P}_1^{*}(s)=\frac{1}{|D(\mathrm{M})|}\left\{\left(1+\frac{\eta}{s}\right)(-\lambda i(s))+\mathrm{P}_4^{*}(s)\mu\left[f(s)(\mu_1+\mu_2)(\mu-\mu_4)\right]\right\}$$

$$=\left(1+\frac{\eta}{s}\right)m_{21}^{*}(s)+\mu\, m_{24}^{*}(s)\mathrm{P}_4^{*}(s) \tag{4.14}$$

$$\mathrm{P}_2^{*}(s)=\frac{1}{|D(\mathrm{M})|}\left\{\left(1+\frac{\eta}{s}\right)(\lambda^2 g_3(s))+\mathrm{P}_4^{*}(s)\mu[\lambda\mu_1-f(s)g_1(s)](\mu-\mu_4)\right\}$$

$$=\left(1+\frac{\eta}{s}\right)m_{31}^{*}(s)+\mu\, m_{34}^{*}(s)\mathrm{P}_4^{*}(s) \tag{4.15}$$

$$\mathrm{P}_3^{*}(s)=\frac{1}{|D(\mathrm{M})|}\left\{\left(1+\frac{\eta}{s}\right)(-\lambda^3)+\mathrm{P}_4^{*}(s)\mu\left[f(s)j(s)-\mu_1\lambda g_2(s)\right]\right\}$$

$$=\left(1+\frac{\eta}{s}\right)m_{41}^{*}(s)+\mu\, m_{44}^{*}(s)\mathrm{P}_4^{*}(s) \tag{4.16}$$

From matrix theory, the characteristic roots s_i , $i=1,2,3,4$ of M provided are all real and distinct. Explore $s_0=0$, it can be obtained by partial fraction decompositions as

$$\frac{(s+\eta)^2}{s}m_{11}^{*}(s)=1+\sum_{k=0}^{4}\frac{\{g_1(s_k)[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)]-\lambda g_3(s_k)(\mu_1+\mu_2)\}(s+\eta)^2}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=1+n_{11}^{*}(s)$$

$$\frac{(s+\eta)^2}{s}m_{21}^{*}(s)=\sum_{k=0}^{4}\frac{-\lambda[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)](s+\eta)^2}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{21}^{*}(s)$$

$$\frac{(s+\eta)^2}{s}m_{31}^{*}(s)=\sum_{k=0}^{4}\frac{\lambda^2 g_3(s_k)(s+\eta)^2}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{31}^{*}(s)$$

$$\frac{(s+\eta)^2}{s}m_{41}^{*}(s)=\sum_{k=0}^{4}\frac{-\lambda^3(s+\eta)^2}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{41}^{*}(s)$$

$$(s+\eta)m_{12}^{*}(s)=\sum_{k=0}^{4}\frac{-\mu_1[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)](s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{12}^{*}(s)$$

$$(s+\eta)m_{22}^{*}(s)=1+\sum_{k=0}^{4}\frac{f(s_k)[g_2(s_k)g_3(s_k)-\lambda(\mu-\mu_4)](s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=1+n_{22}^{*}(s)$$

$$(s+\eta)m_{32}^{*}(s)=\sum_{k=0}^{4}\frac{-\lambda f(s_k)g_3(s_k)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{32}^{*}(s)$$

$$(s+\eta)m_{42}^{*}(s)=\sum_{k=0}^{4}\frac{\lambda^{2} f(s_k)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{42}^{*}(s)$$

$$(s+\eta)m_{13}^{*}(s)=\sum_{k=0}^{4}\frac{\mu_1 g_3(s_k)(\mu_1+\mu_2)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{13}^{*}(s)$$

$$(s+\eta)m_{23}^{*}(s)=\sum_{k=0}^{4}\frac{-f(s_k)g_3(s_k)(\mu_1+\mu_2)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{23}^{*}(s)$$

$$(s+\eta)m_{33}^{*}(s)=1+\sum_{k=0}^{4}\frac{[f(s_k)g_1(s_k)-\lambda\mu_1]g_3(s_k)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=1+n_{33}^{*}(s)$$

$$(s+\eta)m_{43}^{*}(s)=\sum_{k=0}^{4}\frac{\lambda[\lambda\mu_1-f(s_k)g_1(s_k)](s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{43}^{*}(s)$$

$$(s+\eta)m_{14}^{*}(s)=\sum_{k=0}^{4}\frac{-\mu_1(\mu_1+\mu_2)(\mu-\mu_4)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{14}^{*}(s)$$

$$(s+\eta)m_{24}^{*}(s)=\sum_{k=0}^{4}\frac{f(s_k)(\mu_1+\mu_2)(\mu-\mu_4)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{24}^{*}(s)$$

$$(s+\eta)m_{34}^{*}(s)=\sum_{k=0}^{4}\frac{[\lambda\mu_1-f(s_k)g_1(s_k)](\mu-\mu_4)(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=n_{34}^{*}(s)$$

and

$$(s+\eta)m_{44}^{*}(s)=1+\sum_{k=0}^{4}\frac{\{f(s_k)[g_1(s_k)g_2(s_k)-\lambda(\mu_1+\mu_2)]-\mu_1\lambda g_2(s_k)\}(s+\eta)}{\prod_{i=1,i\neq k}^{4}(s_k-s_i)(s-s_k)}=1+n_{44}^{*}(s)$$

where $n_{ij}^{*}(s)'s$, denote the summation terms in the above expressions.

Applying these in equation 4.12 and after some algebraic manipulations, we will get

$$\mathrm{P}_4^{*}(s)=\frac{\frac{2}{\alpha^2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]\left(\frac{\eta}{s}-\sum_{j=1}^{4}n_{j1}^{*}(s)\right)}{1+\frac{2}{\alpha^2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]\mu\sum_{j=1}^{4}n_{j4}^{*}(s)}$$

which implies $\mathrm{P}_4^{*}(s)=\frac{2}{\alpha^2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]\left(\frac{\eta}{s}-d_1^{*}(s)\right)\left[1+\frac{2}{\alpha^2}\left[\Omega-\sqrt{\Omega^2-\alpha^2}\right]\mu d_4^{*}(s)\right]^{-1}$ (4.17)

where $d_i^{*}(s)=\sum_{j=1}^{4}n_{ji}^{*}(s),\quad i=1,4.$

The previous equation can be expressed as,

$$\mathrm{P}_4^*(s) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{\alpha}\right)^{n+1}\left[\frac{\eta}{s}\left(\frac{\Omega-\sqrt{\Omega^2-\alpha^2}}{\alpha}\right)^{n+1} - d_1^*(s)\left(\frac{\Omega-\sqrt{\Omega^2-\alpha^2}}{\alpha}\right)^{n+1}\right]\left[\mu d_4^*(s)\right]^n \qquad (4.18)$$

Taking inversion on equations 2.26-2.29 and doing some algebraic operations, we get

$$\mathrm{P}_0(t) = m_{11}(t) + \eta\int_o^t m_{11}(u)du + \int_o^t \mu\, m_{14}(t-u)\mathrm{P}_4(u)du \qquad (4.19)$$

$$\mathrm{P}_1(t) = m_{21}(t) + \eta\int_o^t m_{21}(u)du + \int_o^t \mu\, m_{24}(t-u)\mathrm{P}_4(u)du \qquad (4.20)$$

$$\mathrm{P}_2(t) = m_{31}(t) + \eta\int_o^t m_{31}(u)du + \int_o^t \mu\, m_{34}(t-u)\mathrm{P}_4(u)du \qquad (4.21)$$

$$\mathrm{P}_3(t) = m_{41}(t) + \eta\int_o^t m_{41}(u)du + \int_o^t \mu\, m_{44}(t-u)\mathrm{P}_4(u)du \qquad (4.22)$$

Therefore, equations 3.13 and 4.18-4.22 completely determine all system size probabilities.

CONCLUSION

In the transient-state analysis, a four heterogeneous server queueing system subject to catastrophes is constructed then the time-dependent probabilities for the number of customers in the system is obtained.

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