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On The Lattice of Subgroups of 3x3 Matrices over Z_2

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ABSTRACT

Let \mathcal{G} be the set of all 3 X 3 non-singular matrices $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, where a,b,c,d,e,f,g,h,i are

integers modulo p. Then \mathcal{G} is a group under matrix multiplication modulo p, of order $(p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$ let G be the subgroup of \mathcal{G} defined by

$G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathcal{G} : \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1 \right\}$. Then G is of order $\frac{(p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})}{p - 1}$. Let

L(G) be the lattice formed by all subgroups G. In this paper, we give the structure of L(G) in the case when sp=2.

KEY WORDS: Lagrange's Theorem, Sylow's Theorem, p-Sylow Subgroup, Lattice of Subgroups

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INTRODUCTION

Let $L(G)$ be the lattice formed by all subgroups of a group G . Study on Lattices of subgroups of groups has quite a long history, starting with Richard Dedekind's work¹⁰ in 1877, including Ada Rottlaender's paper⁴⁷ from 1928 and later numbers important contributions by Reinhold Baer, Qystein Ore, Kenkichi Iwasawa, Leonid Efimovich Sadovskii, Michio Suzuki, Giovanni Zaher, Mario Curzio, Federico Menegazzo, Roland Schmidt, Stewart Stonehewer, Giorgio Busetto, and many-many others.

A celebrated theorem of O. Ore¹² in 1938 states that "If $L(G)$ is a distributive lattice, any finite set of elements from G generates a cyclic subgroup and vice-versa." Thereafter, subgroup lattice theory has witnessed many contributions namely O. Ore¹², R. Baer¹, K. Iwasawa⁸, A. W. Jones⁹, Michio Suzuki^{11, 2, 15} etc.,

In 1992 Karen M. Gragg and P. S. Kung⁶ have attempted to characterize the finite groups with a consistent lattice of subgroups. In that endeavour, they discovered that the lattice of subnormal subgroups of a finite group is consistent and dually semimodular (lower semimodular). A. Vethamanickam has cited from their theorem and has given a counter example in his thesis¹⁶. Suzuki's¹¹ results are mainly concerning L -isomorphic groups. i.e, groups whose lattices of subgroups are isomorphic.

Our original attempt was to study some lattice theoretic properties of $L(G)$ where G is the group of 3×3 matrices whose determinant is 1 modulo p , where p is a prime. In this paper we give the structure of $L(G)$ when $p = 2$. In section 1, we give the preliminary definitions needed for the development of the paper and a lemma for finding the order of G .

1. PRELIMINARIES

Definition 1.1:

A partial order on a non-empty set P is a binary relation \leq on P that is reflexive, antisymmetric and transitive. The pair (P, \leq) is called a partially ordered set or poset. A poset. (P, \leq) is totally ordered if every $x, y \in P$ are comparable, that is $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by \leq .

Definition 1.2:

Let (P, \leq) be a poset and let $S \subseteq P$. An upper bound for S is an element $x \in P$ for which $s \leq x \forall s \in S$. The least upper bound of S is called the supremum or join of S . A lower bound of S is an element $x \in P$ for which $x \leq s \forall s \in S$. The greatest lower bound of S is called the infimum or meet of S . Poset (P, \leq) is called a lattice if every pair $x, y \in P$ has a supremum and an infimum.

Definition 1.3: Interval

For $a, b \in L$, $a \leq b$, we define the intervals:

The closed interval $[a, b] = \{x: a \leq x \leq b\}$.

The half – open intervals

$(a, b] = \{x: a < x \leq b\}$

$[a, b) = \{x: a \leq x < b\}$

The open interval

$(a, b) = \{x: a < x < b\}$

Definition 1.4: Semimodular lattice

A lattice L is called semi modular if whenever a covers $a \wedge b$, then $a \vee b$ covers b , for all $a, b \in L$.

Definition 1.5: Join- irreducible element

An element a of a lattice L is called join-irreducible if $x \vee y = a$ implies either $x=a$ or $y=a$.

Definition 1.6: Consistent Lattice

A Lattice L is said to be consistent if whenever j is a join-irreducible element in L , then for every $x \in L$, $x \vee j$ is join-irreducible in the upper interval $[x, 1]$.

Theorem 1.7: Lagrange’s Theorem

If G is a finite group and H is a subgroup of G , then order of H is a divisor of order of G .

Definition 1.8:

If G is a group and $a \in G$, then order of a is the least positive integer m such that $a^m = e$.

Result 1.9:

If G is a finite group and $a \in G$, then order of a is a divisor of order of G .

Result 1.10:

If H is a non empty finite subset of a group G and H is closed under the binary operation in G , then H is a subgroup of G .

Theorem 1.11: Sylow’s Theorem

If p is a prime number and $p^x || |G|$, then G has a subgroup of order p^x . If $p^m || |G|$, and $p^{m+1} \nmid |G|$, then G has a subgroup of order p^m .

Definition 1.12: p-Sylow Subgroup

A subgroup of G of order p^m , where $p^m || |G|$, but $p^{m+1} \nmid |G|$, is called a p -Sylow subgroup of G .

Theorem 1.13: Sylow's Theorem

The number of p-sylow subgroups in G, for a given prime, is of the form 1+kp. In particular, this number is a divisor of order of G, that is $1+kp \mid |G|$.

Definition 1.14:

A subgroup N of G is said to be a normal subgroup of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Result 1.15:

If H is the only subgroup of order H in the finite group G, then H is a normal subgroup of G.

Theorem 1.16:

Let G be a group of order pq, where p and q are distinct primes and $p < q$. then G has only one subgroup of order q. This subgroup of order q is normal in G.

Result 1.17:

If N is a normal subgroup of G and H is any subgroup of G, then NH is a subgroup of G.

Result 1.18:

If N and M are two normal subgroup of G, then NM is also a normal subgroup of G.

Definition 1.19: (Kernel of the Homomorphism)

Let $f: G \rightarrow G'$ be a homomorphism. The Kernel of f, (denoted by Kerf) is defined by $\ker f = \{x \in G: f(x) = e'\}$ where e' is the identity of G' .

Theorem 1.20(First Isomorphism Theorem)

If $f: G \rightarrow G'$ be an onto homomorphism with $K = \ker f$, then $\frac{G}{K} \cong G'$. In other words, every homomorphic image of a group G is isomorphic to a quotient group of G.

Definition 1.21: (General Linear Group)

The set of nxn Matrices over Z_p with non zero determinant forms a non abelian group under matrix multiplication and is called the General Linear Group, denoted by $GL(n, Z_p)$.

Definition 1.22: (Special Linear Group)

The set of nxn Matrices over Z_p with determinant value 1 forms a non abelian group under matrix multiplication and is called the Special Linear Group, denoted by $SL(n, Z_p)$.

We first prove the following:

Lemma 1.21:

$$\text{Let } G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a, b, c, d, e, f, g, h, i \in Z_p, \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0 \right\}$$

G is a group under matrix multiplication modulo p .

$$\text{Let } G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in G : \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1 \right\}.$$

$$\text{Then } |G| = \frac{(p^n-1)(p^n-p)(p^n-p^2)\dots(p^n-p^{n-1})}{p-1}.$$

Proof:

We first prove that $|G| = (p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$

We will count the $n \times n$ matrices whose rows are linearly independent. We do so by building up a matrix from scratch. The first row can be anything other than the zero row so there are $p^n - 1$ possibilities. The second row must be linearly independent from the first, which is to say that it must not be a multiple of the first. Since there are p multiples of the first row, there are $p^n - p$ possibilities for the second row in general, the i^{th} row must be linearly independent from the first $i-1$ rows, which means that it cannot be a linear combinations of the first $i-1$ rows. There are p^{i-1} linear combinations of the first $i-1$ rows, so there are $p^n - p^{i-1}$ possibilities for the i^{th} row. Once we build the entire matrix this way, we know that the rows are all linearly independent by choice. Also, we can build any $n \times n$ matrix whose rows are linearly independent in this fashion. Thus there are $(p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$ matrices.

Next we claim that $|G| = \frac{(p^n-1)(p^n-p)(p^n-p^2)\dots(p^n-p^{n-1})}{p-1}$

Consider the homomorphism $\det: GL(n, Z_p) \rightarrow Z_p - \{0\}$. This map is Surjective; that is, the image of $GL(n, Z_p)$ under \det is the whole space $Z_p - \{0\}$. This is true because, for instance the matrix

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is an invertible $n \times n$ matrix of determinant a . Since $SL(n, Z_p)$ is the Kernel of the homomorphism, it follows from the first Isomorphism theorem that $GL(n, Z_p) \setminus SL(n, Z_p) \cong Z_p - \{0\}$.

Thus, we have proved $|G| = \frac{(p^n-1)(p^n-p)(p^n-p^2)\dots(p^n-p^{n-1})}{p-1}$.

2. IN THIS SECTION, WE ARRANGE THE ELEMENTS OF G ACCORDING TO THEIR ORDERS:

Let G be the set of all 3×3 non-singular matrices over Z_2 . Thus G is a group under matrix multiplication modulo 2 and

$$|G| = (p^n - 1)(p^n - p)(p^n - p^2) \dots (p^n - p^{n-1})$$

$$\begin{aligned}
 &= (2^3 - 1)(2^3 - 2)(2^3 - 2^2) \\
 &= (8 - 1)(8 - 2)(8 - 4) \\
 &= (7)(6)(4) \\
 &= 168
 \end{aligned}$$

Let G be the subgroup of \mathcal{G} defined by

$$G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathcal{G}; \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1 \right\}$$

$$\begin{aligned}
 \text{Then } |G| &= \frac{(p^n-1)(p^n-p)(p^n-p^2)\dots(p^n-p^{n-1})}{p-1} \\
 &= \frac{(2^3-1)(2^3-2)(2^3-2^2)}{2-1} \\
 &= \frac{(8-1)(8-2)(8-4)}{1} \\
 &= (7)(6)(4) \\
 &= 168
 \end{aligned}$$

2.1 Element Of Order 1(One Element)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.2 Elements Of Order 2 (21 Elements)

$$\begin{aligned}
 &\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
 &\cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
 &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

2.3 Elements Of Order 3 (56 Elements)

$$\begin{aligned}
 &\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
 & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
 & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\
 & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
 \end{aligned}$$

3. IN THIS SECTION, WE FIND ALL THE SUBGROUPS OF G OF DIFFERENT ORDERS:

According to Lagrange’s theorem. We need to check only among the divisors of 168 for the orders of the subgroups.

3.1 Subgroups Of Order 2

Let H be an arbitrary subgroups of G of order 2. Then the elements of H must have order 1 or 2.

Thus, all the subgroups of G of order 2 are obtained as follows:

$$\begin{aligned}
 H_1 &= \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\}, H_2 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\}, H_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}, \\
 H_4 &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\}, H_5 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\}, H_6 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}, \\
 H_7 &= \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_8 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_9 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, \\
 H_{10} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, H_{11} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_{12} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, \\
 H_{13} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, H_{14} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, H_{15} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, \\
 H_{16} &= \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_{17} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_{18} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},
 \end{aligned}$$

$$H_{19} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_{20} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, H_{21} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}$$

3.2. Subgroups Of Order 3

Since 3 is a prime number, any subgroup of G of order 3 is cyclic and hence it is generated by an element of order 3.

Thus, all the subgroups of G of order 3 are obtained as follows:

$$K_1 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, K_2 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_3 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\}, K_4 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_5 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, K_6 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_7 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, K_8 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_9 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, K_{10} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_{11} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, K_{12} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_{13} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, K_{14} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_{15} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, K_{16} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\},$$

$$K_{17} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, K_{18} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_{19} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, K_{20} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$K_{21} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, K_{22} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$K_{23} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\}, K_{24} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}$$

$$K_{25} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, K_{26} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$K_{27} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, K_{28} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}$$

Since $|G| = 2^3 \times 3 \times 7, 3 \nmid |G|$ but $3^2 \nmid |G|$.

Therefore, G has a 3 – sylow subgroup of order 3.

The number of 3-sylow subgroups is of the form $1+3k$ and $1+3k \mid |G|$

Therefore, $1 + 3k \mid 2^3 \times 7$.

The possible values for k are 0,1,2 and 9.

Therefore, the maximum number of 3-sylow subgroups of G of order 3 is 28 when k=9.

So, these are the only subgroups of order 3.

3.3 Subgroups Of Order 4

Let L be an arbitrary subgroup of G of order 4. Then the elements of L must have orders 1,2 or 4. If L contains an element of order 4, then L is generated by an element of order 4. Thus, all the subgroups of G of order 4 are obtained as follows:

$$L_1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\}, L_2 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$L_3 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\}, L_4 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$L_5 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, L_6 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\},$$

$$L_7 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, L_8 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$L_9 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}, L_{10} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\},$$

$$L_{11} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, L_{12} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \right\},$$

$$L_{13} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, L_{14} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\},$$

$$\begin{aligned}
 L_{15} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, L_{16} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, \\
 L_{17} &= \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, L_{18} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, \\
 L_{19} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, L_{20} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, \\
 L_{21} &= \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, L_{22} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, \\
 L_{23} &= \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, L_{24} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, \\
 L_{25} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, L_{26} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}, \\
 L_{27} &= \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}, L_{28} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, \\
 L_{29} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}, L_{30} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}, \\
 L_{31} &= \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}, L_{32} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, \\
 L_{33} &= \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, L_{34} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}, \\
 L_{35} &= \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}.
 \end{aligned}$$

Since first twenty one subgroups of order 4 contains exactly two elements of order 4 and we have only forty two elements of order 4 and first twenty one sub groups of order 2 and next fourteen subgroups of order 4 contains exactly 3 elements of order 2, there will be no other subgroups of order 4 except the above thirty five.

Here,

$$\begin{aligned}
 H_{19} \subset L_1; H_{15} \subset L_2; H_{20} \subset L_3; H_5 \subset L_4; H_4 \subset L_5; H_2 \subset L_6; H_6 \subset L_7; H_{10} \subset L_8; H_{17} \subset L_9; \\
 H_{12} \subset L_{10}; H_3 \subset L_{11}; H_1 \subset L_{12}; H_{14} \subset L_{13}; H_{18} \subset L_{14}; H_9 \subset L_{15}; H_{21} \subset L_{16}; H_{13} \subset L_{17}; H_{11} \\
 \subset L_{18};
 \end{aligned}$$

$$H_8 \subset L_{19}; H_7 \subset L_{20}; H_{16} \subset L_{21}; H_1, H_2, H_{15} \subset L_{22}; H_1, H_5, H_{19} \subset L_{23}; H_2, H_4, H_{20} \subset L_{24};$$

$$H_3, H_4, H_{10} \subset L_{25}; H_5, H_{12}, H_6 \subset L_{26}; H_6, H_{17}, H_3 \subset L_{27}; H_7, H_{21}, H_{20} \subset L_{28}; H_8, H_{10}, H_9$$

$$\subset L_{29};$$

$$H_9, H_{13}, H_{14} \subset L_{30}; H_{11}, H_{15}, H_{13} \subset L_{31}; H_{11}, H_{15}, H_{17} \subset L_{32}; H_{12}, H_{14}, H_7 \subset L_{33};$$

$$H_{13}, H_{19}, H_3 \subset L_{34}; H_{21}, H_{15}, H_{13} \subset L_{35}.$$

3.4 Subgroups of Order 6

Let M be an arbitrary subgroup of G of ordered 6. Since $|M| = 2 \times 3$, by sylow's theorem M has exactly one subgroup of order 3.

Thus, all the subgroups of G of order 6 are obtained as follows:

$$M_1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_2 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$M_4 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$M_5 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$M_6 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$M_7 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_8 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$M_9 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_{10} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_{25} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_{26} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_{27} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$M_{28} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}.$$

Here,

$$H_1, H_{10}, K_1, H_{21} \subset M_1; H_1, H_{14}, K_2, H_{17} \subset M_2; H_3, H_7, K_3, H_1 \subset M_3; H_{12}, H_{18}, K_4, H_2 \subset M_4;$$

$$H_{14}, H_{16}, K_5, H_2 \subset M_5; H_6, H_8, K_6, H_2 \subset M_6; H_2, H_9, K_7, H_{17} \subset M_7; H_4, H_{11}, K_8, H_5 \subset M_8;$$

$$H_4, H_{12}, K_9, H_{16} \subset M_9; H_3, H_{13}, K_{10}, H_{18} \subset M_{10}; H_3, H_{14}, K_{11}, H_{19} \subset M_{11}; H_4, H_{14}, K_{12}, H_{18} \subset M_{12};$$

$$H_{15}, H_{21}, K_{13}, H_3 \subset M_{13}; H_5, H_{13}, K_{14}, H_{20} \subset M_{14}; H_5, H_{21}, K_{15}, H_9 \subset M_{15}; H_5, H_{10}, K_{16}, H_{16} \subset M_{16};$$

$$H_9, H_{20}, K_{17}, H_6 \subset M_{17}; H_{13}, H_{21}, K_{18}, H_6 \subset M_{18}; H_{15}, H_{18}, K_{19}, H_6 \subset M_{19}; H_7, H_{11}, K_{20}, H_8 \subset M_{20};$$

$$H_7, H_{15}, K_{21}, H_{10} \subset M_{21}; H_{11}, H_{12}, K_{22}, H_{10} \subset M_{22}; H_{13}, H_{19}, K_{23}, H_4 \subset M_{23};$$

$$H_{17}, H_{19}, K_{24}, H_7 \subset M_{24}; H_{19}, H_{20}, K_{25}, H_{11} \subset M_{25}; H_{17}, H_{20}, K_{26}, H_8 \subset M_{26};$$

$$H_{12}, H_{15}, K_{27}, H_8 \subset M_{27}; H_1, H_{16}, K_{28}, H_9 \subset M_{28}.$$

Since each subgroup of order 6 contains two elements of order 3 and we have only 56 elements of order 3 and three elements of order 2 and we have only 21 elements of order 2, there will be no other subgroups of order 6 except the above twenty eight.

3.5 Subgroups Of Order 7

Since $|G| = 2^3 \times 3 \times 7$, $7 \mid |G|$ but $7^2 \nmid |G|$.

Therefore, G has a 7 – sylow subgroup of order 7.

The number of 7-sylow subgroups is of the form $1+7k$ and $1+7k \mid |G|$

Therefore, $1 + 7K \mid 2^3 \times 3$.

The possible values for K are 0 and 1.

Therefore, the maximum number of 7-sylow subgroups of G of order 7 is 8 when k=1.

So, these are the only subgroups of order 7.

Thus, all the subgroups of G of order 7 are obtained as follows:

$$N_1 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}$$

$$N_2 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}$$

$$N_3 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \right\}$$

$$N_4 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}$$

$$N_5 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}$$

$$N_6 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, e \right\}$$

$$N_7 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, e \right\}$$

$$N_8 = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\}$$

3.6 Subgroups Of Order 8

Since $|G| = 2^3 \times 3 \times 7$, $2^3 || |G|$ but $2^4 \nmid |G|$.

Therefore, G has a 2 – sylow subgroups of order 8.

The number of 2-sylow subgroups is of the form $1+2k$ and $1+2k \mid |G|$

Therefore, $1 + 2K \mid 3 \times 7$.

The possible values for K are 0 ,1 and 10.

Therefore, the maximum number of 2-sylow subgroups of G of order 8 is 21 when k=10.

Since G has no element of order 8, the elements of subgroups of order 8 must have order 1,2 or 4.

Thus, all the subgroups of G of order 8 are obtained as follows:

$$P_1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\}$$

$$P_2 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\}$$

$$P_{17} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$P_{18} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$P_{19} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$P_{20} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$P_{21} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$L_1, H_1, H_5, H_{18}, H_8 \subset P_1; L_2, H_{11}, H_{13}, H_1, H_2 \subset P_2; L_3, H_2, H_4, H_7, H_{21} \subset P_3;$

$L_4, H_1, H_{19}, H_{12}, H_6 \subset P_4; L_5, H_2, H_{20}, H_3, H_{10} \subset P_5; L_6, H_1, H_{15}, H_4, H_{20} \subset P_6;$

$L_7, H_{17}, H_3, H_5, H_{12} \subset P_7; L_8, H_3, H_4, H_8, H_9 \subset P_8; L_9, H_{11}, H_{16}, H_6, H_3 \subset P_9;$

$L_{10}, H_5, H_6, H_{14}, H_7 \subset P_{10}; L_{11}, H_4, H_{10}, H_6, H_{17} \subset P_{11}; L_{12}, H_2, H_{15}, H_5, H_{19} \subset P_{12};$

$L_{13}, H_{12}, H_7, H_9, H_{13} \subset P_{13}; L_{14}, H_9, H_{14}, H_{11}, H_{15} \subset P_{14}; L_{15}, H_8, H_{10}, H_{13}, H_{14} \subset P_{15};$

$L_{16}, H_7, H_{20}, H_{16}, H_{18} \subset P_{16}; L_{17}, H_{19}, H_8, H_{21}, H_{16} \subset P_{17}; L_{18}, H_{15}, H_{13}, H_{16}, H_{17} \subset P_{18};$

$L_{19}, H_{10}, H_9, H_{18}, H_{19} \subset P_{19}; L_{20}, H_{12}, H_{14}, H_{21}, H_{20} \subset P_{20}; L_{21}, H_{11}, H_{17}, H_{21}, H_{18} \subset P_{21};$

3.7 Subgroups Of Order 12

Let Q be an arbitrary subgroup of order 12. Since $|Q| = 3 \times 4$, by multiplying a subgroup of order 3 and a subgroup of order 4, that is, by finding $K_i L_j$ for all i, j, we get a subset of more than 12 elements or elements of order 7, which cannot exist in a subgroup of order 12.

Hence we get the conclusion that a subgroup of order 12 cannot exist.

Subgroups Of Order 14

Let R be an arbitrary subgroup of order 14. Since $|R| = 2 \times 7$, by multiplying a subgroup of order 2 and a subgroup of order 7, that is, by finding $H_i N_j$ for all i, j, we get in each case elements of order 3 or order 4, which cannot exist in a subgroup of order 14.

Hence we get the conclusion that a subgroup of order 14 cannot exist.

3.8 Subgroups Of Order 21

Let S be an arbitrary subgroup of order 21. Since $|S| = 7 \times 3$, the number of 7-sylow subgroups of order 7 in S is $1+7K$ and $1+7K/3$. The possible value of K is 0 only.

$$S_8 = \left\{ \begin{array}{l} \left(\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right), \\ \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right), e \end{array} \right\}$$

Here

$$N_1, K_4, K_8, K_{11}, K_{18}, K_{21}, K_{26}, K_{28} \subset S_1;$$

$$N_2, K_{28}, K_{13}, K_{12}, K_6, K_{14}, K_{22}, K_{24} \subset S_2;$$

$$N_3, K_{27}, K_{25}, K_{12}, K_{18}, K_3, K_{16}, K_7 \subset S_3;$$

$$N_4, K_{27}, K_{17}, K_8, K_1, K_{10}, K_5, K_{24} \subset S_4;$$

$$N_5, K_{26}, K_{23}, K_5, K_{19}, K_{22}, K_3, K_{15} \subset S_5;$$

$$N_6, K_{25}, K_{21}, K_{15}, K_{10}, K_2, K_6, K_9 \subset S_6;$$

$$N_7, K_{23}, K_{20}, K_4, K_{13}, K_{17}, K_{16}, K_2 \subset S_7;$$

$$N_8, K_1, K_7, K_9, K_{11}, K_{14}, K_{19}, K_{20} \subset S_8;$$

3.9 Subgroups of Order 24:

Let T be an arbitrary subgroup of order 24. Since $|T| = 3 \times 8$, by multiplying a subgroup of order 3 and a subgroup of order 8, that is, by finding $K_i P_j$ for all i, j, we get a subset of more than 24 elements, which cannot exist in a subgroup of order 24.

Hence we get the conclusion that a subgroup of order 24 cannot exist.

3.10 Subgroups Of Order 28:

Let U be an arbitrary subgroup of order 28. Since $|U| = 4 \times 7$, by multiplying a subgroup of order 4 and a subgroup of order 7, that is, by finding $L_i N_j$ for all i, j, we get in each case element of order 3, which cannot exist in a subgroup of order 28. Hence we get the conclusion that a subgroup of order 28 cannot exist.

3.11 Subgroups Of Order 42

Let V be an arbitrary subgroup of order 42. Since $|V| = 2 \times 3 \times 7$, by multiplying a subgroup of order 2 and a subgroup of order 21, that is, by finding $H_i S_j$ for all i, j, we get a subset of more than 42 elements, which cannot exist in a subgroup of order 42. Hence we get the conclusion that a subgroup of order 42 cannot exist.

3.12 Subgroups of Order 56

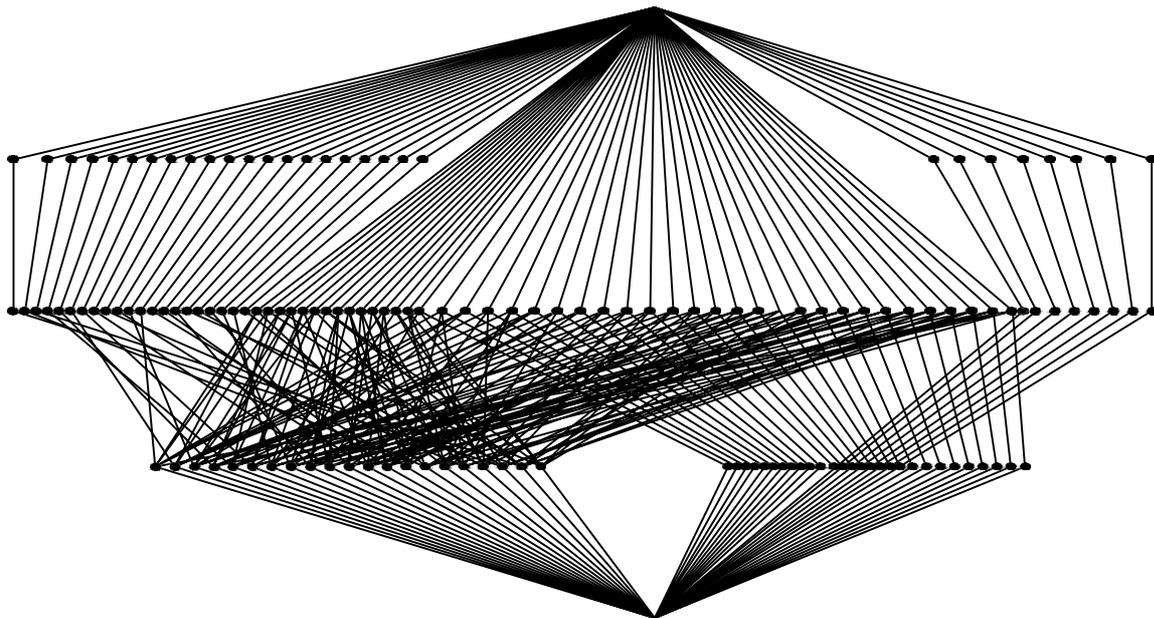
Let W be an arbitrary subgroup of order 56. Since $|W| = 7 \times 8$, by multiplying a subgroup of order 7 and a subgroup of order 8, that is, by finding $N_i P_j$ for all i, j , we get in each case element of order 3, which cannot exist in a subgroup of order 56. Hence we get the conclusion that a subgroup of order 56 cannot exist.

3.13 Subgroups of Order 84

Let X be an arbitrary subgroup of order 84. Since $|X| = 4 \times 21$, by multiplying a subgroup of order 4 and a subgroup of order 21, that is, by finding $L_i S_j$ for all i, j , we get a subset of more than 84 elements, which cannot exist in a subgroup of order 84. Hence we get the conclusion that a subgroup of order 84 cannot exist.

4. THE STRUCTURE OF $L(G)$

According to the above results, we have the diagram of the Lattice of subgroups of $M_3(\mathbb{Z}_2)$ whose elements have determinant values 1.



I Row (Left to Right) : H_1, H_2, \dots, H_{21} and K_1, K_2, \dots, K_{28} .

II Row (Left to Right) : L_1, L_2, \dots, L_{35} and M_1, M_2, \dots, M_{28} and N_1, N_2, \dots, N_8 .

III Row (Left to Right) : P_1, P_2, \dots, P_{21} and S_1, S_2, \dots, S_8

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