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Numerical Solution of First Order Ordinary Differential Equations

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ABSTRACT

In this article, we are presenting numerical solutions of first order differential equations arising in various applications of science and engineering using some classical numerical methods. We are considering only such practical problems which contain differential equations of the first order. Picard's and Taylor series methods are used for solving such type of problems.

KEYWORDS: First order differential equations, Picard's method, Taylor series method, Numerical examples.

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INTRODUCTION:

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc. and as such play a very important role in all modern scientific and engineering studies. For applied mathematics, there are three phases for the study of a differential equation:

(a) Formulation of differential equation from the given physical situation, called modelling.

(b) Solutions of these differential equations by evaluating the arbitrary constants from the given conditions.

(c) Physical interpretation of the solution.

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. In applied mathematics, every geometric or physical problem when translated into mathematical symbols gives rise to a differential equation. There are many classical numerical schemes which are used for solving linear aswell as nonlineardifferential equations. Some common techniques are Picard method, Euler method, Taylor series method, finite difference method, finite element method, finite volume method, spectral method, Runge-Kutta method, etc.The Taylor series algorithm is one of the earliest algorithms for the approximate solution of initial value problems for ordinary differential equations. Newton[6] used it in his calculation and Euler[7] describes it in his work. Since then one can find many mentions of it such as Liouville[8], Peano[9], Picard[10]. In Yang & Liu[2], Picard iterative technique is used for solving initial value problems of singular fractional differential equations. Picard method is used for solving ordinary differential equations in Hirayama[11]. In Corliss & Chang[12], Taylor series method is used for solving differential equations.

PICARD'S ITERATIVE METHOD:

A number of numerical methods are available for the solution of first order differential equations of the form

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx, \tag{2}$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx.$$

This is an integral equation equivalent to (1), for it contain the unknown y under the integral sign. As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx.$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx.$$

Continuing this process, a sequence of functions of x i.e., $y_1, y_2, y_3, \ldots \ldots ..$ is obtained each giving a better approximation of the desired solution than the preceding one.

TAYLOR'S SERIES METHOD:

Consider the first order differential equation (1). Differentiating (1), we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx},$$

i.e.

$$y'' = f_x + f_y f \tag{3}$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + - - - - \tag{4}$$

gives the values of y for every value of x for which (4) converges.

On finding the values y_1 for $x = x_1$ from (4), y', y'' can be evaluated at $x = x_1$ by mean of (1), (3) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (4).

APPLICATION OF DIFFERENTIAL EQUATIONS:

In this section, some numerical experiments are performed for solving some applications of first order differential equations using some classical numerical methods. Numerical data show the accuracy of the proposed numerical methods.

RESISTED MOTION:

Suppose a moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are displacement and velocity of the particle at that instant(Assume the particle starts from rest). The equation of motion of the particle is

$$v\frac{dv}{dx} = -cx - bv^2, \tag{5}$$

with initial condition $v(0) = k$. Equation (5) is nonlinear in v. It is difficult to find the solution of nonlinear differential equations in comparison to linear differential equations. We convert nonlinear differential equations into linear differential equations by using some substitutions.

Put $v^2 = z$ and $2v\frac{dv}{dx} = \frac{dz}{dx}$ in (5), we get

$$\frac{dz}{dx} + 2bz = -2cx, \tag{6}$$

with initial condition $z(0) = r$. Comparing (6) with (1), we get

$$f(x,z) = -2bz - 2cx = -(2bz + 2cx)$$

By Picard's Iterative Method:

Integrating (6) between limits, we get

$$\int_{z_0}^{z} dz = \int_{x_0}^{x} f(x,z)dx, \tag{7}$$

$$z = z_0 - \int_{x_0}^{x} (2bz + 2cx)dx.$$

For a first approximation z_1 to the solution, we put $z = z_0$ in $f(x,z)$ and integrate (7), we get

$$z_1 = z_0 - \int_{x_0}^{x} (2bz_0 + 2cx)dx.$$

For a second approximation z_2, we put $z = z_1$ in $f(x,z)$ and integrate (7), we get

$$z_2 = z_0 - \int_{x_0}^{x} (2bz_1 + 2cx)dx.$$

Continuing this process, a sequence of functions of x i.e., $z_1, z_2, z_3, \ldots \ldots ..$ is obtained each giving a better approximation of the desired solution than the preceding one. The numerical solution of (5) is obtained from the relation $v^2 = z$.

By Taylor's Series Method:

Consider the first order differential equation (6). Differentiating (6), we get

$$\frac{d^2z}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{dz}{dx},$$

i.e.

$$z'' = f_x + f_z f \tag{8}$$

Differentiating this successively, we can get z''', z^{iv} etc. Putting $x = x_0$ and $z = 0$, the values of $(z')_0$, $(z'')_0$, $(z''')_0$ can be obtained. Therefore the Taylor's series

$$z(x) = z_0 + (x - x_0)(z')_0 + \frac{(x - x_0)^2}{2!}(z'')_0 + \frac{(x - x_0)^3}{3!}(z''')_0 + - - - - \tag{9}$$

gives the values of z for every value of x for which (9) converges.

On finding the values z_1 for $x = x_1$ from (9), z', z'' can be evaluated at $x = x_1$. Then z can be expanded about $x = x_1$. The numerical solution of (5) is obtained from the relation $v^2 = z$.

Numerical Observations:

We are discussing two cases:

Case I:

For $b = c = 1/2$, Equation (6) becomes

$$\frac{dz}{dx} + z = -x, \tag{10}$$

with initial condition $z(0) = 0$. The exact solution of such problem is

$$z(x) = (1 - e^{-x}) - x$$

<u>For Picard's Iterative Method:</u>

$$z_1 = -\frac{x^2}{2},$$

$$z_2 = -\frac{x^2}{2} + \frac{x^3}{6},$$

$$z_3 = -\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24},$$

$$z_4 = -\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{120},$$

$$z_5 = -\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720},$$

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For Taylor Series Method:

$$z' = -x - z \quad \Longrightarrow \quad z'(0) = -z(0) = 0$$

$$z'' = -1 - z' \quad \Longrightarrow \quad z''(0) = -1 - z'(0) = -1$$

$$z''' = -z'' \quad \Longrightarrow \quad z'''(0) = -z''(0) = 1$$

$$z'''' = -z''' \quad \Longrightarrow \quad z''''(0) = -z'''(0) = -1$$

Taylor series expansion is

$$z(x) = z(0) + xz'(0) + \frac{x^2}{2}z''(0) + \frac{x^3}{6}z'''(0) + \frac{x^4}{24}z''''(0) + - - - -$$

$$z(x) = -\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + - - - -$$

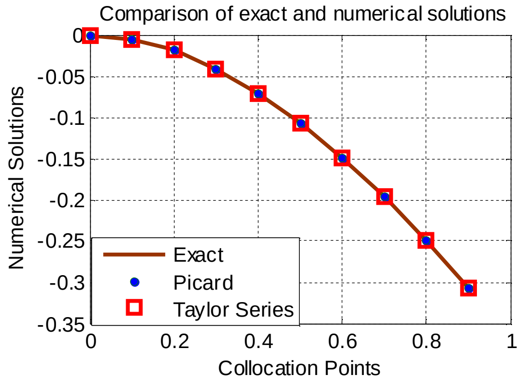

Figure 1Comparison of exact and numerical solutions of Example 1.

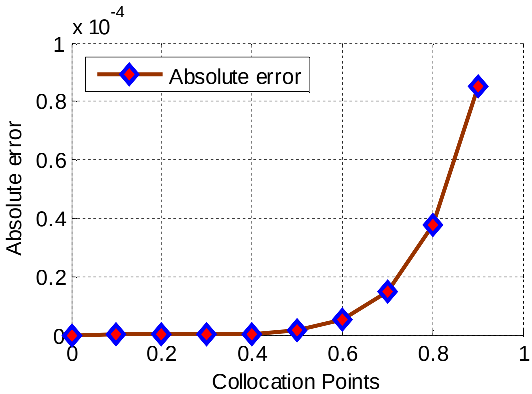

Figure 2 indicate the absolute errors for Example 1

Figure[1] and Figure[2] shows the comparison of exact and numerical solutions of Example 1 for $b = c = 1/2$ (Taking first six terms for Picard's and Taylor's series method).

Case II:

For $b = \frac{3}{2}, c = 1,$ Equation (6) becomes

$$\frac{dz}{dx} + 3z = -2x, \tag{11}$$

with initial condition $z(0) = 0$. The exact solution of such problem is

$$z(x) = \frac{2}{9}(1 - e^{-3x}) - \frac{2}{3}x$$

For Picard's Iterative Method:

$$z_1 = -x^2,$$

$$z_2 = -x^2 + x^3,$$

$$z_3 = -x^2 + x^3 - \frac{3}{4}x^4,$$

$$z_4 = -x^2 + x^3 - \frac{3}{4}x^4 + \frac{9}{20}x^5,$$

$$z_5 = -x^2 + x^3 - \frac{3}{4}x^4 + \frac{9}{20}x^5 - \frac{27}{120}x^6,$$

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For Taylor Series Method:

$$z' = -2x - 3z \quad \Rightarrow \quad z'(0) = -3z(0) = 0,$$

$$z'' = -2 - 3z' \quad \Rightarrow \quad z''(0) = -2 - 3z'(0) = -2,$$

$$z''' = -3z'' \quad \Rightarrow \quad z'''(0) = -3z''(0) = 6,$$

$$z'''' = -3z''' \quad \Rightarrow \quad z''''(0) = -3z'''(0) = -18,$$

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Taylor series expansion is

$$z(x) = z(0) + xz'(0) + \frac{x^2}{2}z''(0) + \frac{x^3}{6}z'''(0) + \frac{x^4}{24}z''''(0) + ----$$

$$z(x) = -x^2 + x^3 - \frac{3}{4}x^4 + \frac{9}{20}x^5 - \frac{27}{120}x^6 + --------$$

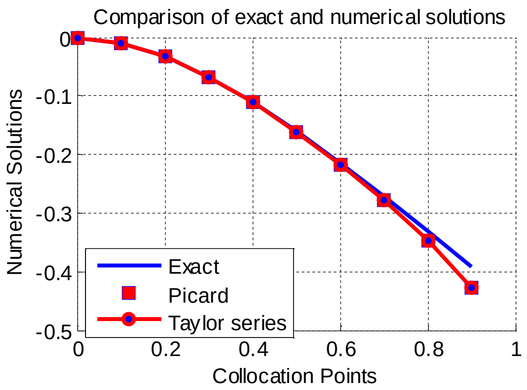

Figure 3 Comparison of exact and numerical solutions of Example 1(Case II)

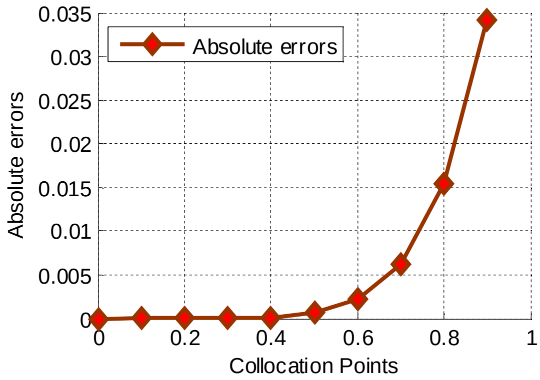

Figure 4Absolute errors for example 1 (Case II)

Figure[3] and Figure[4] shows the comparison of exact and numerical solutions of Example 1for$b = \frac{3}{2}$, $c = 1$ (Taking first six terms for Picard's and Taylor's series method).

ATMOSPHERIC PRESSURE:

To find the atmospheric pressure $p\ lb. per\ ft.$ at a height $z\ ft.$ above the sea level, when the temperature is constant.

Let a vertical column of air of unit cross-section. Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height$z + \delta z$. Let ρ be the density at a height$z$. Since the thin column δz of air is being pressured upwards with pressure p and downwards with pressure $p + \delta p$. By considering equilibrium, we get

$$p = p + \delta p + g\rho\delta z \tag{12}$$

From (12), we get

$$\frac{dp}{dz} = -g\rho, \tag{13}$$

which is the differential equation giving the atmospheric pressure at a height z.

When the temperature is constant, $p = k\rho$ using Boyle's law, we get

$$\frac{dp}{dz} = -g\frac{p}{k}, \tag{14}$$

with initial condition $p(0) = p_0$. The exact solution is

$$p = p_0 e^{-\frac{gz}{k}}.$$

Letting $p_0 = 1$.

For Picard's Iterative Method:

$$p_1 = 1 - \frac{gz}{k},$$

$$p_2 = 1 - \frac{gz}{k} + \frac{1}{2!}\left(\frac{gz}{k}\right)^2,$$

$$p_3 = 1 - \frac{gz}{k} + \frac{1}{2!}\left(\frac{gz}{k}\right)^2 - \frac{1}{3!}\left(\frac{gz}{k}\right)^3,$$

$$p_4 = 1 - \frac{gz}{k} + \frac{1}{2!}\left(\frac{gz}{k}\right)^2 - \frac{1}{3!}\left(\frac{gz}{k}\right)^3 + \frac{1}{4!}\left(\frac{gz}{k}\right)^4,$$

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For Taylor series method:

$$p' = -\frac{g}{k}p, \qquad p'(0) = -\frac{g}{k},$$

$$p'' = -\frac{g}{k}p', \qquad p''(0) = \frac{g^2}{k^2},$$

$$p''' = -\frac{g}{k}p'', \qquad p'''(0) = -\frac{g^3}{k^3},$$

$$p'''' = -\frac{g}{k}p''', \qquad p''''(0) = \frac{g^4}{k^4},$$

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Using Taylor's series method,

$$p(z) = p(0) + zp'(0) + \frac{z^2}{2!}p''(0) + \frac{z^3}{3!}p'''(0) + \frac{z^4}{4!}p''''(0) - - - - -$$

$$p(z) = 1 - z\frac{g}{k} + \frac{z^2}{2!}\frac{g^2}{k^2} - \frac{z^3}{3!}\frac{g^3}{k^3} + \frac{z^4}{4!}\frac{g^4}{k^4} - - - - - -$$

CONCLUSION:

Picard's and Taylor's series methods are powerful mathematical tools for solving linear and nonlinear differential equations. It is concluded that Picard's and Taylor's series methods gives more accurate solutions, which are much closer to exact solutions, for solving first order differential equations arising in some applications of sciences and engineering.

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