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Investigation of Inter Relation Among Different Types of Continuous Functions on Convex Topological Space

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ABSTRACT

In this paper some continuous functions has been introduced using both the topology τ and convexity $\mathcal{C}$ on the same underlying set X where $(X, \tau, \mathcal{C})$ is termed as convex topological space and inter relation among them are also investigated .

AMS SUBJECT CLASSIFICATION : 52A01 , 54C10 .

KEYWORDS : Convex topological space , $\delta_\star - \mathcal{C}$ continuous function , $\theta_\star - \mathcal{C}$ continuous function , regular $\mathcal{C}$- continuous function .

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1. INTRODUCTION

The development of "abstract convexity" has emanated from different sources in different ways ; the first type of development basically banked on generalization of particular problems such as separation of convex *sets* [1] , *extremality* [2,3] or continuous *selection* [4] . The second type of development lay before the reader such axiomatizations , which in every case of design , express particular point of view of convexity . With the view point of generalized topology which enters into convexity via the closure or hull operator , Schmidt and Hammer, , introduced some axioms to explain abstract convexity . The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birchoff and Frink , Schmidt, Hammer .

The axiomatizations as proposed by M.L.J. Van De Vel in his paper Theory of Convex *Structure* [5] will be followed Through out in this paper .

The author has discussed in " Topology and Convexity on the same *set* [6] " and introduced the compatibility of the topology with a convexity on the same underlying set . At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet $(X, \tau, \mathcal{C})$ and call it convex topological space only to bring back "compatibility" in another way subsequently . With this compatibility , Van De Vel has called the triplet $(X, \tau, \mathcal{C})$ a topological convex structure .

In this paper , Art. 2 deals with some early definitions , results and in Art. 3 we have discussed mainly inter relation among different types of continuous functions .

2. PREREQUISITES :

Definition 2.1 [6] **:** Let X be a non empty set . A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\phi, X \in \mathcal{C}$
2. $\mathcal{C}$ is stable for intersection , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty , then $\cap \mathcal{D} \in \mathcal{C}$
3. $\mathcal{C}$ is stable for nested unions , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty and totally ordered by set inclusion , then $\cup \mathcal{D} \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure . The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets .

Definition 2.2 [6] **:** Let $\mathcal{C}$ be a convexity on set X . Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by $co(A) = \cap \{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3 [6] **:** Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X . The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y; called the relative convexity of Y.

Note 2.4 [6] **:** The hull operator co_Y of a subspace $(Y, \mathcal{C}_Y)$ satisfy the following :

$\forall A \subseteq Y : co_Y(A) = co(A) \cap Y$.

Definition 2.5 [6]**:** Let $(X, \mathcal{C})$ be a convex structure and let τ be a topology on X . Then τ is said to be compatible with the convex structure $(X, \mathcal{C})$ if all polytopes of $\mathcal{C}$ are closed in τ where polytopes means convex hull of a finite set . Also the triplet $(X, \tau, \mathcal{C})$ is then called topological convex structure .

Note 2.6 [6] **:** Let $(X, \tau, \mathcal{C})$ be a topological convex structure . Then collection of all closed sets in (X, τ) are subset of $\mathcal{C}$.

Definition 2.7 [7] **:** Let (X, τ) be a topological space and let $\mathcal{C}$ be a convexity on X . Then the triplet $(X, \tau, \mathcal{C})$ is called a convex topological space (CTS in short) .

Definition 2.8 [8] **:** Let $(X, \tau, \mathcal{C})$ be a convex topological space . A set $P \subseteq X$ is said to be C-regular open if $P = int(co(P))$.

Result 2.9 [8] **:** Let A be a subset of a convex topological space $(X, \tau, \mathcal{C})$. Then $int(co(A))$ is a C- regular open set .

Note 2.10 [8] **:** In a convex topological space $(X, \tau, \mathcal{C})$ for any subset A of X , the set $int(co(A))$ is a C- regular open set . Also a subset B of X is called C- regular closed set if its complement is C- regular open set.

Definition 2.11 [8] **:** [8] Let $(X, \tau, \mathcal{C})$ be a convex topological space . Let S be a subset of X and $x \in X$.

(a) x is called $\delta_\star - \mathcal{C}$ cluster point of S if $S \cap int(co(U)) \neq \emptyset$, for each open nbd. U of x .

(b) The family of all $\delta_\star - \mathcal{C}$ cluster points of S is called the $\delta_\star - \mathcal{C}$ closure of S and is denoted by $[S]_{\delta_\star}$.

(c) A subset S of X is called $\delta_\star - \mathcal{C}$ closed if $[S]_{\delta_\star} = S$.

The complement of a $\delta_\star - \mathcal{C}$ closed set is said to be a $\delta_\star - \mathcal{C}$ open set .

Definition 2.12 [8] **:** Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is said to be $\delta_\star - \mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f\left(int(co(U))\right) \subseteq int(co(V))$.

Result 2.13 [8] **:** A function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is $\delta_\star - \mathcal{C}$ continuous iff for each $x \in X$ and each each C- regular open set V containing $f(x)$, there exists a C- regular open set U containing x such that $f(U) \subseteq V$.

3. COMPARISON OF DIFFERENT TYPES OF CONTINUOUS FUNCTIONS :

Definition 3.1 : Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces . A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is said to be strongly $\theta_\star - \mathcal{C}$ continuous , $\theta_\star - \mathcal{C}$ continuous , regular $\mathcal{C}$- continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f(co(U)) \subseteq V$, $f(co(U)) \subseteq co(V)$, $f(U) \subseteq int(co(V))$ respectively .

Remark 3.2 : For any convex topological space we have $P \subseteq co(P)$. This shows that strongly $\theta_\star - \mathcal{C}$ continuous $\Rightarrow$ $\theta_\star - \mathcal{C}$ continuous **.**

The following example shows that the converse of the above implication may not be true in general

Example 3.3 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X\}$ and f is the identity mapping I_X on X .

Here f is $\theta_\star - \mathcal{C}$ continuous on X. Also for $a \in X$ if we consider the open nbd. $\{a\}$ of $f(a)$, then there is no $U \in \tau$ such that $f(co(U)) \subseteq \{a\}$. So f is not strongly $\theta_\star - \mathcal{C}$ continuous on X .

Note 3.4 : We now show that $\theta_\star - \mathcal{C}$ continuous and regular $\mathcal{C}$- continuous are two independent concepts which follows from the next two examples .

Example 3.5 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\mathcal{C}_1 = \{\emptyset, X, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a, b\}\}$ and f is the identity mapping I_X on X .

Here f is $\theta_\star - \mathcal{C}$ continuous but not regular $\mathcal{C}$- continuous on .

Example 3.6 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a\}\}$ and f is the identity mapping I_X on X .

Here f is regular $\mathcal{C}$- continuous but not $\theta_\star - \mathcal{C}$ continuous on .

Theorem 3.7 : Let a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be $\theta_\star - \mathcal{C}$ continuous and open . Then f is regular $\mathcal{C}$- continuous function .

Proof : Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is $\theta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f(co(U)) \subseteq co(V)$. Thus $f(U) \subseteq f\left(co((U))\right) \subseteq co(V)$. Now U is an open set and f is open mapping . Thus $f(U)$ is an open set in Y which is contained in $co(V)$. So $f(U) \subseteq int(co(V))$. Consequently f is regular $\mathcal{C}$- continuous function .

Theorem 3.8 : (a) If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ is strongly $\theta_\star - \mathcal{C}$ continuous and $g : (Y, \sigma, \mathcal{C}_2) \to (Z, \gamma, \mathcal{C}_3)$ is regular $\mathcal{C}$- continuous , then $g \circ f : (X, \tau, \mathcal{C}_1) \to (Z, \gamma, \mathcal{C}_3)$ is $\delta_\star - \mathcal{C}$ continuous .

(b) The following implications hold :

strongly $\theta_\star - \mathcal{C}$ continuous $\Rightarrow \delta_\star - \mathcal{C}$ continuous $\Rightarrow$ regular $\mathcal{C}$- continuous .

Proof : (a) Let $x \in X$ and W be any open set containing $(g \circ f)(x)$. Since g is regular $\mathcal{C}$- continuous , there exists an open nbd. V of $f(x)$ in Y such that $g(V) \subseteq int(co(W))$. Again since f is strongly $\theta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x in X such that $f(co(U)) \subseteq V$. Now $f\left(int(co(U))\right) \subseteq f(co(U)) \subseteq V \Rightarrow g\left(f\left(int(co(U))\right) \subseteq\right) g\left(f(co(U))\right) \subseteq g(V) \subseteq int(co(W)) \Rightarrow$ $(g \circ f)\left(int(co(U))\right) \subseteq int(co(W))$. This shows that $g \circ f$ is $\delta_\star - \mathcal{C}$ continuous .

(b) Let f be strongly $\theta_\star - \mathcal{C}$ continuous . Also let $x \in X$ and V be any open nbd. of $f(x)$. Then there exists an open nbd. U of x in X such that $f(co(U)) \subseteq V \Rightarrow f\left(int(co(U))\right) \subseteq f(co(U)) \subseteq V = int(V) \subseteq int(co(V))$. Hence f is $\delta_\star - \mathcal{C}$ continuous .

Again let f be $\delta_\star - \mathcal{C}$ continuous . Also let $x \in X$ and V be any open nbd. of $f(x)$ in Y. Then there exists an open nbd. U of x in X such that $f\left(int(co(U))\right) \subseteq int(co(V))$. Let $W = int(co(U))$. Then W is open nbd. of x such that $f(W) \subseteq int(co(V))$. Thus f is regular $\mathcal{C}$- continuous .

Note 3.9 : The following examples show that none of these implications in the above theorem is reversible .

Example 3.10 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \to (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{b\}\}$, $\mathcal{C}_2 = \{\emptyset, X\}$ and f is the identity mapping I_X on X .

Here f is $\delta_\star - \mathcal{C}$ continuous but not strongly $\theta_\star - \mathcal{C}$ continuous on .

Example 3.11 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \to (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a\}\}$ and f is the identity mapping I_X on X .

Here f is regular $\mathcal{C}$- continuous but not $\delta_\star - \mathcal{C}$ continuous on X .

Definition 3.12 : A convex topological space $(X, \tau, \mathcal{C})$ is said to be an semi $\mathcal{C}$- regular space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int(co(U)) \subseteq V$.

Theorem 3.13 : For a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ the following properties are true :

(a) If Y is an semi $\mathcal{C}$- regular space and f is $\delta_\star - \mathcal{C}$ continuous , then f is continuous .

(b) If X is an semi $\mathcal{C}$- regular space and f is regular $\mathcal{C}$- continuous , then f is $\delta_\star - \mathcal{C}$ continuous .

Proof : (a) Let Y be an semi $\mathcal{C}$- regular space and $x \in X$. Then for each open nbd. V of $f(x)$, there exists an open nbd. W of $f(x)$ such that $f(x) \in W \subseteq int(co(W)) \subseteq V$. Since f is $\delta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f\Big(int(co(U))\Big) \subseteq int(co(W))$. Since U is an open set , $f(U) = f(int(U)) \subseteq f\Big(int(co(U))\Big) \subseteq int(co(W)) \subseteq V$ *i.e.*, $f(U) \subseteq V$. Hence f is continuous .

(b) Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is regular $\mathcal{C}$- continuous , there exists an open nbd. U of x such that $f(U) \subseteq int(co(V))$. Again since X is an semi $\mathcal{C}$- regular space there exists an open nbd. W of x such that $int(co(W)) \subseteq U$. Thus $f\Big(int(co(W))\Big) \subseteq f(U) \subseteq int(co(V))$. Hence f is $\delta_\star - \mathcal{C}$ continuous .

Corollary 3.14 : If $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ are semi $\mathcal{C}$- regular spaces , then the concepts on a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$, $\delta_\star - \mathcal{C}$ continuity , continuity , regular $\mathcal{C}$- continuity are equivalent .

Definition 3.15 : A CTS $(X, \tau, \mathcal{C})$ is said to be an almost $\mathcal{C}$- regular if for each $\mathcal{C}$- regular closed set F and each $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 3.16 : Let $(X, \tau, \mathcal{C})$ be an almost $\mathcal{C}$- regular space where τ is compatible with $\mathcal{C}$. Then for each $x \in X$ and each $\mathcal{C}$- regular open nbd. V of x , there exists a $\mathcal{C}$- regular open nbd. W of x such that $x \in W \subseteq co(W) \subseteq V$.

Proof : Let $x \in X$ and V be a $\mathcal{C}$- regular open set containing x . Then $x \notin X \setminus V$ and $X \setminus V$ is a $\mathcal{C}$- regular closed set . Thus there exist disjoint open sets U_1 , U_2 such that $x \in U_1$ and $X \setminus V \subseteq U_2$. Now $U_1 \cap U_2 = \emptyset \Rightarrow cl(U_1) \cap U_2 = \emptyset \Rightarrow co(U_1) \cap U_2 = \emptyset$ [Since τ is compatible with $\mathcal{C}$] $\Rightarrow co(U_1) \subseteq X \setminus U_2 \subseteq V \Rightarrow int(co(U_1)) \subseteq V$. Let $W = int(co(U_1))$. Then $x \in W$ and W is a $\mathcal{C}$- regular open set . Also $W = int(co(U_1)) \subseteq co(U_1) \Rightarrow co(W) \subseteq co(U_1) \subseteq \mathrm{V} \Rightarrow \mathrm{x} \in \mathrm{W} \subseteq \mathrm{co}(\mathrm{W}) \subseteq \mathrm{V}$.

Theorem 3.17 : For a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ the following hold :

(1) If Y is almost $\mathcal{C}$- regular space where σ is compatible with $\mathcal{C}_2$ and f is $\theta_\star - \mathcal{C}$ continuous , then f is $\delta_\star - \mathcal{C}$ continuous .

(2) If X is almost $\mathcal{C}$- regular space where τ is compatible with $\mathcal{C}_1$, Y is semi $\mathcal{C}$- regular space and f is $\delta_\star - \mathcal{C}$ continuous , then f is strongly $\theta_\star - \mathcal{C}$ continuous .

Proof : 1) Let $x \in X$ and V be a $\mathcal{C}$- regular open nbd. of $f(x)$. Then Y being almost $\mathcal{C}$- regular space , there exist a $\mathcal{C}$- regular open nbd. U of $f(x)$ such that $f(x) \in U \subseteq co(U) \subseteq V$. Since f is $\theta_\star - \mathcal{C}$ continuous , there exists an open nbd. W of x such that $f(co(W)) \subseteq co(U)$. Thus $\left(int(co(W))\right) \subseteq f(co(W)) \subseteq co(U) \subseteq V$. Hence f is $\delta_\star - \mathcal{C}$ continuous .

2) Let $x \in X$ and V be an open nbd. of $f(x)$. Since Y is semi $\mathcal{C}$- regular space , there exists an open nbd. U of $f(x)$ such that $f(x) \in U \subseteq int(co(U)) \subseteq V$. Again by the $\delta_\star - \mathcal{C}$ continuity of f , there exists an open nbd. W of x such that $f\left(int(co(W))\right) \subseteq int(co(U))$. Now $int(co(W))$ is a $\mathcal{C}$- regular open set in X which is almost $\mathcal{C}$- regular . So there is $\mathcal{C}$- regular open nbd. P of x such that $x \in P \subseteq co(P) \subseteq int(co(W))$ .This implies that $f(co(P)) \subseteq f\left(int(co(W))\right) \subseteq int(co(U)) \subseteq V$. Thus f is strongly $\theta_\star - \mathcal{C}$ continuous .

Definition 3.18 : A function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ is called $\mathcal{C}$- regular open if for each $\mathcal{C}$- regular open set U of X , $f(U)$ is open in Y .

Theorem 3.19 : Let a function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ be $\theta_\star - \mathcal{C}$ continuous and $\mathcal{C}$- regular open . Then f is $\delta_\star - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is $\theta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f(co(U)) \subseteq co(V)$. Thus $f\left(int(co(U))\right) \subseteq f\left(co((U))\right) \subseteq co(V)$. Now $\left(int(co(U))\right)$ is a $\mathcal{C}$- regular open set and f is $\mathcal{C}$- regular open mapping . Thus $f\left(int(co(U))\right)$ is an open set in Y which is contained in $co(V)$. So $f\left(int(co(U))\right) \subseteq int(co(V))$. Consequently f is $\delta_\star - \mathcal{C}$ continuous function .

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