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An Integral Representation of Bicomplex Dirichlet Series

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ABSTRACT

In this paper, we have defined the **Bicomplex Dirichlet Series** $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ and investigate its region of convergence. We have also obtained an integral representation of **Bicomplex Dirichlet Series** $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$.

KEYWORDS: Bicomplex numbers, Bicomplex Gamma Function, Bicomplex Riemann Zeta Function, Complex Dirichlet Series, Euler Product

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1. INTRODUCTION

The set of Bicomplex Numbers defined as:

$$C_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0,\ i_1 \neq i_2 \text{ and } i_1^2 = i_2^2 = -1,\ i_1 i_2 = i_2 i_1\}$$

Throughout this paper, the sets of complex and real numbers are denoted by C_1 and C_0, respectively. For details of the theory of Bicomplex numbers, we refer to [1, 2, 3, 4]. We shall use the notations $C(i_1)$ and $C(i_2)$ for the following sets: $C(i_1) = \{u + i_1 v : u, v \in C_0\}$; $C(i_2) = \{\alpha + i_2\beta : \alpha, \beta \in C_0\}$

1.1 Idempotent Elements:

Besides 0 and 1, there are exactly two non – trivial idempotent elements in C_2, denoted as e_1 and e_2 and defined as $e_1 = \frac{1 + i_1i_2}{2}$ and $e_2 = \frac{1 - i_1i_2}{2}$. Note that $e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$.

1.2 Cartesian Idempotent Set:

$$C_2 = C(i_1) \times_e C(i_1) = C(i_1)e_1 + C(i_1)e_2 = \{\xi \in C_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in C(i_1) \times C(i_1)\}$$
$$C_2 = C(i_2) \times_e C(i_2) = C(i_2)e_1 + C(i_2)e_2 = \{\xi \in C_2 : \xi = \xi_1 e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in C(i_2) \times C(i_2)\}$$

1.3 Idempotent Representation Of Bicomplex Numbers:

(I) $C(i_1)$- idempotent representation of Bicomplex Number Throughout this paper $C(i_1)$-idempotent representation of Bicomplex Number is given by

$$\xi = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4) = z_1 + i_2 z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2$$
$$= [(x_1 + x_4) + i_1(x_2 - x_3)]e_1 + [(x_1 - x_4) + i_1(x_2 + x_3)]e_2 = {}^1\xi e_1 + {}^2\xi e_2$$

(II) $C(i_2)$- idempotent representation of Bicomplex Number Throughout this paper $C(i_2)$-idempotent representation of Bicomplex Number is given by

$$\xi = (x_1 + i_2x_3) + i_1(x_2 + i_2x_4) = w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2$$
$$= [(x_1 + x_4) - i_2(x_2 - x_3)]e_1 + [(x_1 - x_4) + i_2(x_2 + x_3)]e_2 = \xi_1 e_1 + \xi_2 e_2$$

1.4 Singular Elements:

Non zero singular elements exist in C_2. In fact, a Bicomplex number $\xi = z_1 + z_2i_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$. Set of all singular elements in C_2 is denoted as O_2.

1.5 Norm:

The norm in C_2 is defined as

$$\|\xi\| = \left\{|z_1|^2 + |z_2|^2\right\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2} = \left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right]^{1/2}$$

C_2 becomes a modified Banach algebra, in the sense that $\xi, \eta \in C_2$, we have, in general,

$$\|\xi.\eta\| \le \sqrt{2}\,\|\xi\|\,\|\eta\|$$

1.6 Complex Dirichlet Series [5, 6, 7]:

In general, a Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \qquad \text{...............} \ \mathbf{(1.1)}$$

where $\{\lambda_n\}$ is a monotonically increasing and unbounded sequence of real numbers, and $s = \sigma + it$ is a complex variable. When the sequence $\{\lambda_n\}$ of exponent is to be emphasized, such a series is called a **Complex Dirichlet series of type** λ_n.

If $\lambda_n = n$, then $f(s)$ is a power series in $z = e^{-s}$. If $\lambda_n = \log n$, then

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \text{...............} \ \mathbf{(1.2)}$$

is called an **Ordinary complex Dirichlet series**.

2. BICOMPLEX DIRICHLET SERIES:

The **Bicomplex Dirichlet series** is defined as

$$f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi} \qquad \text{...} \ \mathbf{(2.1)}$$

where $\{\alpha_n\}$ is a sequence of bicomplex numbers, $\{\lambda_n\}$ is a strictly monotonically increasing and unbounded sequence of positive real numbers and $\xi \in C_2$ is a bicomplex variable. If $\lambda_n = n$, then $f(\xi) = \sum_{n=1}^{\infty} \alpha_n (e^{-\xi})^n$ is a **power series** in $e^{-\xi}$. If $\lambda_n = \log n$, then

$$f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi} \qquad \text{...} \ \mathbf{(2.2)}$$

is a **Ordinary Bicomplex Dirichlet Series**.

If $\alpha_n = 1$ in equation **(3.2)** $f(\xi) = \sum_{n=1}^{\infty} n^{-\xi}$ represent **Bicomplex Riemann Zeta Function**[8, 9, 10, 11] in that consequence we named $f(\xi) = \sum_{n=1}^{\infty} \alpha_n n^{-\xi}$ a **Generalized Bicomplex Riemann Zeta Function**[12, 13, 14].

Note that,

$$\alpha_n e^{-\lambda_n \xi} = ({}^1\alpha_n e^{-\lambda_n {}^1\xi}) e_1 + ({}^2\alpha_n e^{-\lambda_n {}^2\xi}) e_2$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \xi} = \left[\sum_{n=1}^{\infty} {}^1\alpha_n e^{-\lambda_n {}^1\xi}\right] e_1 + \left[\sum_{n=1}^{\infty} {}^2\alpha_n e^{-\lambda_n {}^2\xi}\right] e_2$$

Now we denote the sum function of the series $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$, $\sum_{n=1}^{\infty}{}^1\alpha_n e^{-\lambda_n {}^1\xi}$ and $\sum_{n=1}^{\infty}{}^2\alpha_n e^{-\lambda_n {}^2\xi}$ by $f(\xi)$, ${}^1f({}^1\xi)$ and ${}^2f({}^2\xi)$ respectively.

Thus $f(\xi) = {}^1f({}^1\xi)e_1 + {}^2f({}^2\xi)e_2$

Then $f(\xi) = \sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ is a Bicomplex Dirichlet series and ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n e^{-\lambda_n {}^1\xi}$, ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n e^{-\lambda_n {}^2\xi}$ are Complex Dirichlet series. Throughout, we denote the abscissae of convergence of ${}^1f({}^1\xi)=\sum_{n=1}^{\infty}{}^1\alpha_n e^{-\lambda_n {}^1\xi}$ and ${}^2f({}^2\xi)=\sum_{n=1}^{\infty}{}^2\alpha_n e^{-\lambda_n {}^2\xi}$ by σ_1 and σ_2, and the abscissae of absolute convergence by $\overline{\sigma}_1$ and $\overline{\sigma}_2$, respectively.

THEOREM 2.1: A Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges for $\xi=\xi_0$ iff $\sum_{n=1}^{\infty}{}^1\alpha_n e^{-\lambda_n {}^1\xi}$ converges for ${}^1\xi={}^1\xi_0$ and $\sum_{n=1}^{\infty}{}^2\alpha_n e^{-\lambda_n {}^2\xi}$ converges for ${}^2\xi={}^2\xi_0$.

THEOREM 2.2: If $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges for $\xi=\xi_0$ then $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges in the region

$$\{\xi\in C_2 : Re({}^1\xi) > Re({}^1\xi_0) \text{ and } Re({}^2\xi) > Re({}^2\xi_0)\}$$

$$= \{\xi\in C_2 : x_1 + x_4 > x_1^0 + x_4^0 \text{ and } x_1 - x_4 > x_1^0 - x_4^0\}$$

or equivalently in the region

$$\{\xi\in C_2 : Re(z_1) > Re(z_1^0) \text{ and } |Im(z_2) - Im(z_2^0)| < Re(z_1) - Re(z_1^0)\}.$$

COROLLARY 2.1: If $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ diverges for $\xi=\xi_0$ then $\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ diverges in the region

$$\{\xi\in C_2 : Re({}^1\xi) < Re({}^1\xi_0) \text{ and } Re({}^2\xi) < Re({}^2\xi_0)\}$$

$$= \{\xi\in C_2 : x_1 + x_4 < x_1^0 + x_4^0 \text{ and } x_1 - x_4 < x_1^0 - x_4^0\}$$

or equivalently in the region

$$\{\xi\in C_2 : Re(z_1) < Re(z_1^0) \text{ and } |Im(z_2) - Im(z_2^0)| > Re(z_1) - Re(z_1^0)\}.$$

THEOREM 2.3: The Bicomplex Dirichlet series $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n \xi}$ converges in the region

$$R = \{\xi\in C_2 : Re({}^1\xi) > \sigma_1 \text{ and } Re({}^2\xi) > \sigma_2\}.$$

3. AN INTEGRAL REPRESENTATION OF BICOMPLEX DIRICHLET SERIES:

DEFINITION 3.1:

Let $[a,b]$ be an interval in C_0. A curve C in C_2 is a mapping $\zeta:[a,b]\to C_2$. The trace of C is the set $\{\,\zeta(t)\in C_2 : t\in[a,b]\,\}$.

THEOREM 3.1[15] : Let $\phi: X\to C_2$ be a continuous function, and let γ be a curve defined by mapping $\zeta:[a,b]\to X$. If γ has continuous derivative $\zeta':[a,b]\to C_2$, then

$$\int_\gamma \phi(\zeta(t))\, d\zeta(t) = \int_a^b \phi\,[\zeta(t)]\,\zeta'(t)\,dt$$

BICOMPLEX INTEGRALS AND THE IDEMPOTENT REPRESENTATION:

Let X be domain in C_2 and let $f: X\to C_2$, $f(\zeta) = {}^1f({}^1\zeta)\,e_1 + {}^2f({}^2\zeta)\,e_2$ be a holomorphic function. Let γ be a curve $\zeta(t) = z_1(t) + i_2 z_2(t),\ a\le t\le b$ whose trace is in X, so that $\zeta(t) = {}^1\zeta(t)e_1 + {}^2\zeta(t)\,e_2$, shows that there are curves γ_1 and γ_2, with traces in X_1 and X_2 respectively, such that

$$\gamma_1 : {}^1\zeta = {}^1\zeta(t) \qquad a\le t\le b$$

$$\gamma_2 : {}^2\zeta = {}^2\zeta(t) \qquad a\le t\le b$$

THEOREM 3.2 [1]: Under the above mentioned notations and hypothesis, integrals of f, f_1 and f_2 exists on curves γ, γ_1 and γ_2 respectively and

$$\int_\gamma f(\zeta)d\zeta = \left[\int_{\gamma_1} {}^1f({}^1\zeta)\,d({}^1\zeta)\right]e_1 + \left[\int_{\gamma_2} {}^2f({}^2\zeta)\,d({}^2\zeta)\right]e_2 .$$

DEFINITION 3.2 [15]:

Let $\xi = {}^1\xi\,e_1 + {}^2\xi\,e_2 \in C_2$, $p = p_1 e_1 + p_2 e_2$, $\quad p_1, p_2 \in C_0^+$.

We define

$$\Gamma_2(\xi) = \int_\gamma e^{-p}\, p^{\xi-1}\,dp$$

Where γ is a four dimensional curve in C_2 and $\gamma_1 \equiv \gamma_1(p_1)$, $\gamma_2 \equiv \gamma_2(p_2)$ are component curves with traces in A_1 and A_2, such that $\gamma = \gamma_1 e_1 + \gamma_2 e_2$.

We have obtained the following result regarding the region of convergence of Bicomplex Gamma function.

THEOREM 3.3: Let $\xi = z_1 + z_2 i_2 \in C_2$ with $\mathrm{Re}({}^1\xi) > 0$ and $\mathrm{Re}({}^2\xi) > 0$ then $\Gamma_2(\xi)$ converges and $\Gamma_2(\xi) = \Gamma({}^1\xi)e_1 + \Gamma({}^2\xi)e_2$.

Moreover, $\{\,\xi\in C_2 : \mathrm{Re}({}^1\xi) > 0 \text{ and } \mathrm{Re}({}^2\xi) > 0\} = \{\,\xi\in C_2 : \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|\,\}$.

PROOF: By **Def. 3.2** and **Th. 3.2**

$$\Gamma_2(\xi) = \int_\gamma e^{-p}\, p^{\xi-1}\, dp$$

$$= \int_\gamma \left(e^{-p_1}\, p_1^{\,^1\xi-1} e_1 + e^{-p_2}\, p_2^{\,\xi-1} e_2 \right)(dp_1 e_1 + dp_2 e_2)$$

$$= \left[\int_0^\infty e^{-p_1}\, p_1^{\,^1\xi-1}\, dp_1 \right] e_1 + \left[\int_0^\infty e^{-p_2}\, p_2^{\,^2\xi-1}\, dp_2 \right] e_2$$

$$= \Gamma(^1\xi) e_1 + \Gamma(^2\xi) e_2$$

Now, from the theory of the Gamma function of a complex variable, it is well known that the series $\Gamma(s)$ converges in the half-plane $\mathrm{Re}(s) > 0$.

Therefore, $\Gamma(^1\xi)$ and $\Gamma(^2\xi)$ converge, respectively, for $\mathrm{Re}(^1\xi) > 0$ and $\mathrm{Re}(^2\xi) > 0$.

Hence, $\Gamma_2(\xi) = \Gamma(^1\xi) e_1 + \Gamma(^2\xi) e_2$ converges on $\{ \xi \in C_2 : \mathrm{Re}(^1\xi) > 0 \text{ and } \mathrm{Re}(^2\xi) > 0 \}$.

Now let, $\xi = {}^1\xi\, e_1 + {}^2\xi e_2 = z_1 + i_2 z_2$ and $z_1 = x_1 + i_1 x_2$, $z_2 = x_3 + i_1 x_4$

$^1\xi = z_1 - i_1 z_2 = x_1 + x_4 + i_1(x_2 - x_3)$ and $^2\xi = z_1 + i_1 z_2 = x_1 - x_4 + i_1(x_2 + x_3)$

$\mathrm{Re}(^1\xi) = x_1 + x_4$ and $\mathrm{Re}(^2\xi) = x_1 - x_4$

Since $\mathrm{Re}(^1\xi) > 0$ and $\mathrm{Re}(^2\xi) > 0$

$\Leftrightarrow x_1 + x_4 > 0$ and $x_1 - x_4 > 0$

$\Leftrightarrow x_1 > -x_4$ and $x_1 > x_4$

$\Leftrightarrow x_1 > |x_4|$

$\Leftrightarrow \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$

Hence, $\{ \xi \in C_2 : \mathrm{Re}(^1\xi) > 0 \text{ and } \mathrm{Re}(^2\xi) > 0 \} = \{ \xi \in C_2 : \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)| \}$.

THEOREM 3.4 [15]**:** Let $\xi = {}^1\xi e_1 + {}^2\xi\, e_2 = z_1 + z_2 i_2 \in C_2$ with $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$. Then

$$\frac{1}{\Gamma_2(\omega)} = \frac{1}{\Gamma(^1\omega)} e_1 + \frac{1}{\Gamma(^2\omega)} e_2 .$$

Let $\mu_n = \log \lambda_n$ and $\xi \in C_2$, $p = p_1 e_1 + p_2 e_2$, $p_1, p_2 \in C_0^+$.

Where γ is a four dimensional curve in C_2 and $\gamma_1 \equiv \gamma_1(p_1)$, $\gamma_2 \equiv \gamma_2(p_2)$ are component curves with traces in A_1 and A_2, such that $\gamma = \gamma_1 e_1 + \gamma_2 e_2$.

THEOREM 3.5: Under the above mentioned notations and hypothesis,

$$\sum \alpha_n e^{-\mu_n \xi} = \frac{1}{\Gamma_2(\xi)} \int_\gamma p^{\xi-1} \left(\sum \alpha_n e^{-\lambda_n p} \right) dp ,$$

provided that $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$ and the series on the left is convergent.

PROOF: Let $\xi = z_1 + i_2 z_2 \in C_2$ such that $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$. Then, by **Th. 3.4**,

$$\frac{1}{\Gamma_2(\xi)} = \frac{1}{\Gamma({}^1\xi)} e_1 + \frac{1}{\Gamma({}^2\xi)} e_2 \qquad \text{... (3.1)}$$

Further due to idempotent techniques,

$$p^{\xi-1} = p_1^{\,{}^1\xi-1} e_1 + p_2^{\,{}^2\xi-1} e_2$$

and $\sum \alpha_n e^{-\lambda_n p} = \left(\sum {}^1\alpha_n e^{-\lambda_n {}^1 p} \right) e_1 + \left(\sum {}^2\alpha_n e^{-\lambda_n {}^2 p} \right) e_2$

Now, $p^{\xi-1}\left(\sum \alpha_n e^{-\lambda_n p}\right) = p_1^{\,{}^1\xi-1}\left(\sum {}^1\alpha_n e^{-\lambda_n {}^1 p} \right) e_1 + p_2^{\,{}^2\xi-1}\left(\sum {}^2\alpha_n e^{-\lambda_n {}^2 p} \right) e_2$

$$\int_\gamma p^{\xi-1}\left(\sum \alpha_n e^{-\lambda_n p}\right) dp$$

$$= \int_\gamma \left\{ p_1^{\,{}^1\xi-1}\left(\sum {}^1\alpha_n e^{-\lambda_n {}^1 p} \right) e_1 + p_2^{\,{}^2\xi-1}\left(\sum {}^2\alpha_n e^{-\lambda_n {}^2 p} \right) e_2 \right\}\{dp_1 e_1 + dp_2 e_2\}$$

$$= \left[\int_0^\infty p_1^{\,{}^1\xi-1}\left(\sum {}^1\alpha_n e^{-\lambda_n p_1} \right) dp_1 \right] e_1 + \left[\int_0^\infty p_2^{\,{}^2\xi-1}\left(\sum {}^2\alpha_n e^{-\lambda_n p_2} \right) dp_2 \right] e_2 \qquad \text{... (3.2)}$$

Now by **(3.1)** and **(3.2)**

$$\frac{1}{\Gamma_2(\xi)} \int_\gamma p^{\xi-1}\left(\sum \alpha_n e^{-\lambda_n p}\right) dp$$

$$= \left[\frac{1}{\Gamma({}^1\xi)} e_1 + \frac{1}{\Gamma({}^2\xi)} e_2 \right]$$

$$= \left[\left[\int_0^\infty p_1^{\,{}^1\xi-1}\left(\sum {}^1\alpha_n e^{-\lambda_n p_1} \right) dp_1 \right] e_1 + \left[\int_0^\infty p_2^{\,{}^2\xi-1}\left(\sum {}^2\alpha_n e^{-\lambda_n p_2} \right) dp_2 \right] e_2 \right]$$

$$= \left[\frac{1}{\Gamma({}^1\xi)} \int_0^\infty p_1^{\,{}^1\xi-1}\left(\sum {}^1\alpha_n e^{-\lambda_n p_1} \right) dp_1 \right] e_1 + \left[\frac{1}{\Gamma({}^2\xi)} \int_0^\infty p_2^{\,{}^2\xi-1}\left(\sum {}^2\alpha_n e^{-\lambda_n p_2} \right) dp_2 \right] e_2$$

$$= \left[\frac{1}{\Gamma({}^1\xi)} \sum {}^1\alpha_n \int_0^\infty p_1^{\,{}^1\xi-1} e^{-\lambda_n p_1}\, dp_1 \right] e_1 + \left[\frac{1}{\Gamma({}^2\xi)} \sum {}^2\alpha_n \int_0^\infty p_2^{\,{}^2\xi-1} e^{-\lambda_n p_2}\, dp_2 \right] e_2$$

$$= \left[\frac{1}{\Gamma({}^1\xi)} \sum \frac{{}^1\alpha_n}{\lambda_n^{{}^1\xi}} \Gamma({}^1\xi) \right] e_1 + \left[\frac{1}{\Gamma({}^2\xi)} \sum \frac{{}^2\alpha_n}{\lambda_n^{{}^2\xi}} \Gamma({}^2\xi) \right] e_2$$

$$= \left[\sum \frac{{}^1\alpha_n}{\lambda_n^{{}^1\xi}} \right] e_1 + \left[\sum \frac{{}^2\alpha_n}{\lambda_n^{{}^2\xi}} \right] e_2 = \sum \frac{\alpha_n}{\lambda_n^{\xi}} = \sum \alpha_n \lambda_n^{-\xi} = \sum \alpha_n e^{-\mu_n \xi}, \quad [\because \mu_n = \log \lambda_n]$$

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