Vijimon Moni et al., IJSRR 2019, 8(1), 1877-1887

Research article **Available online www.ijsrr.org** **ISSN: 2279–0543**

International Journal of Scientific Research and Reviews

On The Upper Open Geodetic Domination Number of a Graph

Vijimon Moni.V[1] and Robinson Chellathurai.S[2]

Register Number-12357,
[1]St. Xavier's Catholic College of Engineering, Chunkankadai-629 013
[2]Scott Christian College, Nagercoil-629 003,India.
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012,
Tamil Nadu, India

ABSTRACT

Let $G = (V, E)$ be a connected graph of order n. A set $S \subseteq V(G)$ is called an open geodetic dominating set of G if S is both open geodetic set and dominating set of G. The minimum cardinality of an open geodetic dominating set of G is called the open geodetic domination number of G and is denoted by $\gamma_{og}(G)$. An open geodetic dominating set of minimum cardinality is called γ_{og}- set of G. An open geodetic dominating set S in a connected graph G is called a minimal open geodetic dominating set of G if no proper subset of S is an open geodetic dominating set of G.The maximum cardinality of a minimalopen geodetic domination set of G is the upper open geodetic domination number of G and is denoted by$\gamma^+_{og}(G)$. A minimal open geodetic dominating set of cardinality $\gamma^+_{og}(G)$is called a γ^+_{og}- set of G. The upper open geodetic dominating number of certain classes of graph are determined. Some general properties satisfied by this concept are studied. For any positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $\gamma_{og}(G) = a$ and$\gamma^+_{og}(G) = b$.

KEYWORDS : Open geodetic number, Open geodetic domination number,upper open geodetic dominating number.

AMS Subject Classification: 05C05, 05C69.

*Corresponding author

Vijimon Moni.V

Register Number-12357, Department of Mathematics

St. Xaviers Catholic College of Engineering,

Chunkankadai-629 003, Tamilnadu. India.

Email:vijimon1983@gmail.com.

Mobile: 8946046108.

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to Harary[10]. The *distance*$d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$*geodesic*. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices$u$ and v. The closed interval consists of x, y and all vertices lying on some $x - y$ geodesic of G[1]. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \cup_{x,y \in S} I[x, y]$ is the closure of S. A set $S \subseteq V(G)$ is called a *geodetic set* if $I[S] = V(G)$. Thus every vertex of G is contained in a geodesicjoining some pair of vertices in S.The minimum cardinality of a geodetic set of G is called the *geodetic nu*mber of G and is denoted by $g(G)$. A geodetic set of minimum cardinality is called g-set of G[2,4,5,6]. $N(v) = \{u \in V(G) : uv \in E(G)\}$is called the *neighborhood* of the vertex v in G. A vertex $v$is an*extreme* vertex of a graph G if $<N(v)>$ is complete. A set of vertices D in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G[3,7]. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a *pendent ed*ge, u a leaf and v a *support v*ertex. Let $L(G)$ be the set of all leaves of a graph G. For any connected graph G, a vertex $v \in V(G)$ is called a *cut vertex* of Gif $V - v$ is no longer connected. A set of vertices S in $G$is called a *geodetic dominatin*g set if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its *geodetic domination number* and is denoted by $\gamma_g(G)$. A geodetic dominating set of size$\gamma_g(G)$ is said to be a γ_g*-set* of G[9,12]. A set S of vertices of a connected graph G is an *open geodetic set* if for each vertex v in G either v is an extreme vertex of G and $v \in S$ or v is an internal vertex of a $x - y$ geodesic for some $x, y \in S$. An *open* geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic numbe*r and is denoted by $og(G)$[14].A Set$S \subseteq V(G)$is called an *open geodetic dominating set* of a connected graph G if S is both open geodetic set and dominating set of G. The minimum cardinality of an open geodetic dominating set of G is called *open geodetic domination number* of G and is denoted by $\gamma_{og}(G)$[13]. An open geodetic dominating set of minimum cardinality is called γ_{og}-set of G.For a cut vertex v in a connected graph G and the component H of $G - v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *bran*ch of G at v. The *middle graph* of a graph $G = (V, E)$ is the graph $M(G) = (V \cup E, E')$, Where $uv \in E'$ if and only if either $u$is a vertex of G and v is an edge of G containing u, or u and v are edges in G having a vertex in common.

The following theorem is used in sequel.

Theorem1.1[13]. Let G be a connected graph of order n. Then

i. every open geodetic dominating set of a graph G contains its extreme vertices.
ii. every end vertex belongs to every open geodetic dominating set of G.
iii. if theset S of extreme vertices of G is a open geodetic dominating set of G, then S is theunique minimum open geodetic dominating set of $G$and$\gamma_{og}(G) = |S|$.

THE UPPER OPEN GEODETIC DOMINATION NUMBER OF A GRAPH

Definition2.1. An open geodetic dominating set S in a connected graph $G$is called a minimal open geodetic dominating set of G if no proper subset of S is an open geodetic dominating set of G. The maximum cardinality of a minimal open geodetic dominating set of G is the upper open geodetic set domination number of G and is denoted by $\gamma_{og}^{+}(G)$. A minimal open geodetic dominating set of cardinality $\gamma_{og}^{+}(G)$ is called a γ_{og}^{+}- set of G.

Example2.2. For the graph G given in Figure 1, $S_1=\{v_1,v_2, v_3,v_6, v_9\}$and $S_2 = \{v_1, v_2, v_3, v_5, v_7,v_9\}$are open geodetic dominating sets of G. It is clear that no proper subsets of S_1 and S_2 are open geodetic dominating set of G and so $S_1$and S_2 are minimal open geodetic dominating sets of G. Hence $\gamma_{og}(G) = 5$ and $\gamma_{og}^{+}(G) = 6$. It is clear that there is no minimal open geodetic dominating set of cardinality greater than 6. Therefore

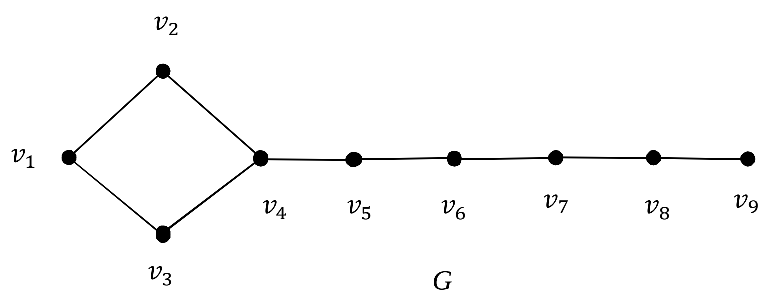


Figure 1
A graph with $\gamma_{og}^{+}(G) = 6$

$\gamma_{og}^{+}(G) = 6$.

Theorem2.3. Let G be a connected graph of order n. Then

(i) every minimal open geodetic dominating set of a graph G contains its extreme vertices. (ii) every end vertex belongs to every minimal open geodetic dominating set of G.

(iii) if$G$ has the unique minimal open geodetic dominating set,then $\gamma_{og}(G)=\gamma_{og}^{+}(G)$.

Proof. (i) Since every minimal open geodetic dominating set of connected graph G is a open geodetic dominating set of G, by Theorem 1.1, (i)and (ii) follows immediately.

(iii)Let S be unique minimal open geodetic dominating set of a connected graph G. Then it is clear that $\gamma_{og}(G) = |S|$ and $\gamma_{og}^{+}(G) = |S|$. Hence $\gamma_{og}(G) = \gamma_{og}^{+}(G)$.

Theorem 2.4. For the complete graph $G = K_n$, $\gamma_{og}^{+}(G) = n$.

Proof. Since every vertex of G is an extreme vertex, then by Theorem 2.3(i) $\gamma_{og}^{+}(G) = n$.

Theorem 2.5. If a connected graph G has m extreme vertices, then $\gamma_{og}^{+}(G) \geq m$.

Proof. As every minimal open geodetic dominating set of a connected graph G contains its extreme vertices, by Theorem 2.3(i)$\gamma_{og}^{+}(G) \geq m$.

Theorem2.6. Let $M(G)$ be the middle graph of a connected graph G of order n.
Then $\gamma_{og}(M(G)) = \gamma_{og}^{+}(M(G)) = n$.

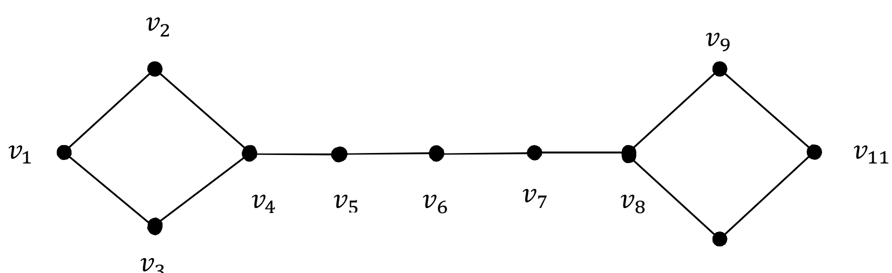
Proof. Let $M(G)$ be the middle graph of a connected graph G of order n. Then it is clear that set of extreme vertices of $M(G)$ is $V(G)$. It is easily verified that $V(G)$ is the unique minimal open geodetic dominating set of $M(G)$. Therefore, by Theorem 2.3(iii)$\gamma_{og}(M(G)) = \gamma_{og}^{+}(M(G)) = n$.

Theorem2.7. Let G be a connected graph of order n, $2 \leq \gamma_{og}(G) \leq \gamma_{og}^{+}(G) \leq n$.

Proof. Since every open geodetic dominating set needs at least two vertices, Therefore $\gamma_{og}(G) \geq 2$. Since every minimal open geodetic dominating set is a open geodeticdominating set of G, $\gamma_{og}(G) \leq \gamma_{og}^{+}(G)$. Also since the set of all vertices of G is an open

geodetic dominating set of G,$\gamma_{og}^{+}(G) \leq n$. Hence$2 \leq \gamma_{og}(G) \leq \gamma_{og}^{+}(G) \leq n$. ■

Remark 2.8. The bounds in Theorem 2.7 are sharp. For the path $G = P_2$, $\gamma_{og}(G) = 2$. For the star $G = K_{1,n-1}$, $\gamma_{og}(G) = \gamma_{og}^{+}(G) = n - 1$. For the complete graph, $G = K_n$,$\gamma_{og}(G) = \gamma_{og}^{+}(G) = n$. Also the bounds in Theorem 2.7 are strict. For the graph G given in Figure 2, $\gamma_{og}(G) = 7$, $\gamma_{og}^{+}(G) = 8$ and $n = 11$. Thus $2 \leq \gamma_{og}(G) \leq \gamma_{og}^{+}(G) \leq n$.



G
Figure 2

Theorem 2.9. For the connected graph $G\gamma_{og}(G) = 2$if and only if $\gamma_{og}^{+}(G) = 2$.

Proof. If$\gamma_{og}^{+}(G) = 2$, then by Theorem 2.7, $\gamma_{og}(G) = 2$. Conversely, let $\gamma_{og}(G) = 2$. Then G contains two extreme vertices u and v such that $S = \{u, v\}$is the uniqueminimum γ_{og}-set of G.

Since S is subset of every open geodetic dominating set itfollows that $S = \{u, v\}$ is the unique minimal open geodetic dominating set of G, sothat$\gamma_{og}^{+}(G) = 2$. ■

Theorem 2.10. Let G be a connected graph of order n.If$\gamma_{og}(G) = n$, if and only if $\gamma_{og}^{+}(G) = n$.

Proof. If$\gamma_{og}(G) = n$, then by Theorem 2.7, $\gamma_{og}^{+}(G) = n$. Conversely, let $\gamma_{og}^{+}(G) = n$. Then $S = V(G)$ is the unique minimal open geodetic dominating set of G. Henceit follows that S is the unique minimum open geodetic dominating set of G, so that$\gamma_{og}(G) = n$. ■

Theorem 2.11. Let G be a connected graph of order n. If $\gamma_{og}(G) = n-1$, then$\gamma_{og}^{+}(G) = n-1$.

Proof. Let$\gamma_{og}(G) = n-1$. Then by Theorem 2.7, $\gamma_{og}^{+}(G) = n$ or $n - 1$. If$\gamma_{og}^{+}(G) = n$, then by Theorem 2.10, $\gamma_{og}(G) = n$, Which is a contradiction. Therefore $\gamma_{og}^{+}(G) = n - 1$.

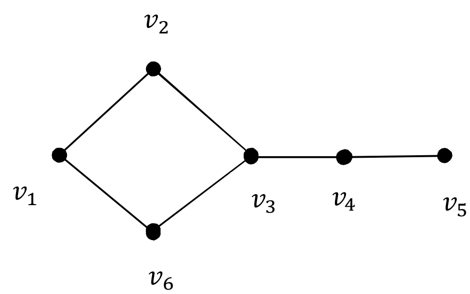
Theorem 2.12. For the complete Bipartite graph G = $K_{m,n}$ with 2 ≤ m ≤ n,$\gamma_{og}^{+}(G) = 4$.

Proof. Let G = $K_{m,n}$.LetX = $\{u_1, u_2, ..., u_m\}$ and Y = $\{v_1, v_2, ..., v_n\}$ be the partitesets of G. Let S = $\{u_i, u_j, v_r, v_s\}$. Then S is a minimal open geodetic dominating setof G and so $\gamma_{og}^{+}(G) \geq 4$. We show that $\gamma_{og}^{+}(G) = 4$. If not, let $\gamma_{og}^{+}(G) \geq 5$. Then thereexists a minimal open geodetic dominating set S'such that $|S'| \geq 5$. If $S' \subseteq$ X,then $S'$is not a open geodetic dominating set of G, Which is a contradiction. If $S' \subseteq Y$, then $S'$is not a open geodetic dominating set of G, Which is a contradiction. Therefore, $S' \subseteq X \cup Y$. Let $S' = S_1 \cup S_2$, Where $S_1 \subseteq X$ and $S_2 \subseteq Y$. Then $|S_1| \geq 2$ and $|S_2| \geq 2$.Since $|S'| \geq 5$, either S_1 or S_2 contains atleast three vertices, withoutloss of generality let us assume that $|S_1| \geq 3$. Let $x, y, z \in S_1$ and , $v \in S_2$. Then$x, y, z, u, v \in S'$. Let $S'' = S' - \{x\}$, Which is a contradiction to S' is a minimal open geodetic dominating set of G. Let $S''=S' - \{x\}$. Then $S''$is a open geodetic dominating set of G such that $S'' \subset S'$ which is a contradiction to S' is a minimal open geodetic dominating set of G.Therefore $\gamma_{og}^{+}(G) = 4$.

Theorem2.13. For any connected non-complete graph G of order n, then $\gamma_{og}^{+}(G) \leq n - \delta(G)$.

Proof. Let S be a upper open geodetic dominating set of a non-complete connectedgraph G order n. Then $\gamma_{og}^{+}(G) = |S|$. We show that $|S| \leq n - \delta(G)$. Let $v \in S$.Assume that v is adjacent to m distinct vertices in S. Since $deg(v) > \delta(G)$, v mustbe adjacent to atleast $\delta(G) - m$ vertices in $V(G) - S$ and so $|V(G) - S| > \delta(G) - m$.If $m = 0$,then $|V(G) - S| \geq \delta(G)$,that is $|S| \leq |V(G)| - \delta(G) = n - \delta(G)$. If $m > 0$, then the m distinct vertices belong to $N[S]$ and doesnot lie on a geodesicjoining any pair of vertices of S, Since S is a minimal open geodetic dominating set of$G$, $|V(G) - S| \geq (\delta(G) - m) + m = \delta(G)$. Hence $|S| \leq n - \delta(G)$. Therefore $\gamma_{og}^{+}(G) \leq n - \delta(G)$. ■

Remark2.14. The bounds in Theorem 2.13 are sharp. For the graph $G = K_{1,n-1}$of order n. It is clear that δ(G) = 1, $n - \delta(G) = n-1$ and $\gamma_{og}^{+}(G) = n-1$. Thus$\gamma_{og}^{+}(G) = n - \delta(G)$. The bounds in Theorem 2.13 can be strict. For the graph G inFigure 3, δ$(G) = 1, \gamma_{og}^{+}(G) = 4, n = 6, n - \delta(G) = 5$.Thus $\gamma_{og}^{+}(G) < n - \delta(G)$.



G
Figure 3

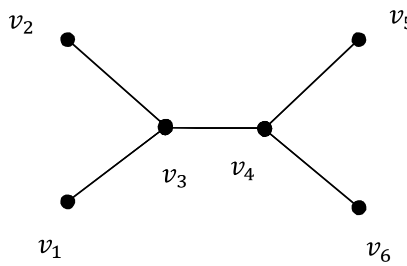
Theorem 2.15. Let G be a connected graph of order n and $u \in V(G)$. If deg(u) = 1, then $\gamma_{og}^{+}(G - u) \leq \gamma_{og}^{+}(G)$.

Proof. Let $u \in V(G)$ and deg(u) = 1. Let S be a minimal open geodetic dominatingset of $G - u$ with maximum cardinality, so $\gamma_{og}^{+}(G - u) = |S|$. Since deg($u$) = 1, u is an end vertex and u is adjacent to exactly one vertex, say v. By Theorem 2.3 every minimal open geodetic dominating set of G contains u. We consider two cases.

case(i): Let $v \in S$. Since S is an open geodetic dominating set of $G - u$, there exists a vertex $w \in V(G - u)$ such that $w \in I[v,x] \subseteq I[S], w \in N[S], v, x \in I[S]$ and $d(v,x) \leq 3$. If $d(v,x) = 3$, then consider the set $S' = (S - \{v\}) \cup \{u,w\}$. If $d(v,x) \leq 2$ then consider the set $S' = (S - \{v\}) \cup \{u\}$. It is straight forward to verify that S' is a minimal open geodetic dominating set of$G$. So that $\gamma_{og}^{+}(G - u) = |S| \leq |S'| \leq \gamma_{og}^{+}(G)$.

case(ii): Let $v \notin S$. Then consider the set $S' = S \cup \{u\}$. It is straight forward toverify that S' is a minimal open geodetic dominating set of G. So that $\gamma_{og}^{+}(G - u) = |S| < |S'| \leq \gamma_{og}^{+}(G)$. Hence in both the cases, $\gamma_{og}^{+}(G - u) \leq \gamma_{og}^{+}(G)$. ■

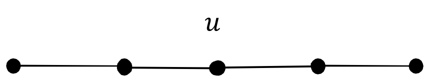
Remark 2.16. The bounds in Theorem 2.15 are sharp. For the graph $G = P_4$,letu be an end vertex of G. It is clear that $\gamma_{og}^{+}(G - u) = 2$ and γ_{og}^{+}(G) = 2. Hence $\gamma_{og}^{+}(G - u) = \gamma_{og}^{+}(G)$. The bounds in Theorem 2.16 can be strict. For the graph G inFigure 4, $\gamma_{og}^{+}(G - u) = 3$ and $\gamma_{og}^{+}(G) = 4$.Hence$\gamma_{og}^{+}(G - u) < \gamma_{og}^{+}(G)$.



G
Figure 4

Remark 2.17. The converse of the Theorem 2.15 is need not true. For the completegraph K_n, it is clear that $\gamma_{og}^+(K_n)$ = n, $\gamma_{og}^+(K_n - u) = n - 1$ and deg(u) = $n - 1$ forevery $u \in V(K_n)$. Hence $\gamma_{og}^+(K_n - u) < \gamma_{og}^+(K_n)$ but deg(u) $\neq$ 1.

Remark 2.18. Theorem 2.15 is not true if deg(u) $\neq$ 1. For the graph $G = P_5$, givenin Figure 5, $\gamma_{og}^+(G)$ = 3, $\gamma_{og}^+(G - u)$ = 4 and deg(u) = 2 $\neq$1.Thus$\gamma_{og}^+(G - u) \nleq \gamma_{og}^+(G)$.



G
Figure 5

Theorem 2.19. For any non-trivial tree T with $n \geq 3$, there exists a vertex $v \in V(T)$ such that $\gamma_{og}^+(T - v) = \gamma_{og}^+(T)$.

Proof. Let T be any non-trivial tree with $n \geq 3$. It can be verified that the result istrue for $n = 3$. Since if $n = 3$ then $T = P_3$. Now consider the case that $n > 3$. Since$T$has atleast one vertex with degree greater than or equal to 2, there exists a vertex$v \in V(T)$ with deg(v) $\geq$ 2 such that v is adjacent to at least one leaf and atmostone non-leaf. If there exists a vertex v such that v is adjacent to atleast one- leafand no non-leaf then it is clear that $T = K_{1,n-1}$ and v is the support vertex So that $\gamma_{og}^+((T - v) = n - 1 = \gamma_{og}^+(T)$.If there does not exist a vertex v such that v is adjacentto exactly one leaf, then it is clear that v is adjacent to two or more leaves. Assumethat v is adjacent to exactly one non-leaf. By Theorem 2.3 every minimal opengeodetic dominating set of T contains its leaves So it is clear that $\gamma_{og}^+(T - v) = \gamma_{og}^+(T)$.If there exists a vertex v such that v is adjacent to exactly one leaf u and one non-leaf, then deg(u) = 1 and deg(v) = 2. Let $T' = T - v - u$. Since deg(u) = 1, By Theorem 2.16, $\gamma_{og}^+(T - v) \leq \gamma_{og}^+(T)$. Hence,$\gamma_{og}^+((T') \leq \gamma_{og}^+(T - u) \leq \gamma_{og}^+(T)$. However,we have $\gamma_{og}^+(T') > \gamma_{og}^+(\mathrm{T}) - 1$. If $\gamma_{og}^+(T') = \gamma_{og}^+(\mathrm{T}) - 1$, then $\gamma_{og}^+(\mathrm{T}) = \gamma_{og}^+(\mathrm{T} - \mathrm{u})$. If$\gamma_{og}^+(T') > \gamma_{og}^+(\mathrm{T}) - 1$, then

$\gamma_{og}^{+}(T') = \gamma_{og}^{+}(T) = \gamma_{og}^{+}(T-u)$. Hence there exists a vertex v ∈ V (T) such that $\gamma_{og}^{+}(T-v) = \gamma_{og}^{+}(T)$.■

Remark 2.20. Theorem 2.19 is not true for any graph G. For the complete graph K_n.

$\gamma_{og}^{+}(K_n - v) \neq \gamma_{og}^{+}(K_n)$ for every $v \in V(K_n)$.

Theorem 2.21. Let G be a connected graph of order n. If $G'$is a graph obtainedby adding k, where $1 \leq k \leq n$, end edges to a graph G, then $\gamma_{og}^{+}(G) \leq \gamma_{og}^{+}(G') \leq \gamma_{og}^{+}(G) + k$.

Proof. Let G be a connected graph of order n and let G'be a connected graphobtained from G by adding k end edges $u_i v_i$ $(1 \leq i \leq k)$, where each $u_i \in V(G)$and $v_i \notin V(G)$.First we show that $\gamma_{og}^{+}(G) \leq \gamma_{og}^{+}(G')$ Let S be a γ_{og}^{+}-set of G, So$\gamma_{og}^{+}(G) = |S|$. We now consider three cases.

Case(i): Let $u_i \in S$ for all i $(1 \leq i \leq k)$. Then let $S' = S \cup \{v_1, v_2, ..., v_k\}$. Since each

$v_i \notin V(G)$is an end vertex of $G'$and $u_i \notin S$,$v_i \notin I[S]$ and $v_i \notin N[S]$, $S'$is a minimal

open geodetic dominating set of G'. Therefore$\gamma_{og}^{+}(G) = |S| < |S'| \leq \gamma_{og}^{+}(G')$.

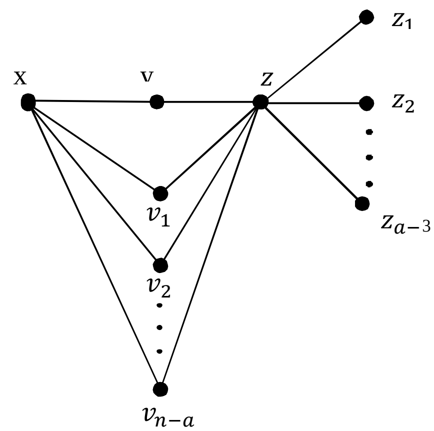
Case(ii): Let $u_i \in S$ for some $i, 1 \leq i \leq k$. Since S is an open geodetic dominating set of G, there exists a vertex $v \notin S$ such that $v \in I[u_i, x] \subseteq I[S]$, $v \in N[S]$ and $d(u_i, x) \leq 3$ for some $x \in S$. If $d(u_i, x) = 3$, then consider the set S' = (S - $\{u_i\}$) ∪ $\{v_i, v\}$. If d(u_i, x) ≤ 2, then consider the set S' = (S - $\{u_i\}$) ∪ $\{v_i\}$. It is easily verified that $S'$is a minimal open geodetic dominating set ofG'. Therefore$\gamma_{og}^{+}(G) = |S| \leq |S'| \leq \gamma_{og}^{+}(G')$.

Case(iii): Let $u_i \in S$ for all $i, 1 \leq i \leq k$.Then by the similar argument as in case(ii),

wecanprove that $\gamma_{og}^{+}(G) \leq \gamma_{og}^{+}(G')$. Next, we show that $\gamma_{og}^{+}(G') \leq \gamma_{og}^{+}(G) + k$.Let $S \subseteq V(G)$and let $S' = S \cup \{v_1, v_2, ..., v_k\}$ be a minimal open geodetic dominatingset of G' with maximum cardinality so that $\gamma_{og}^{+}(G') = |S'| = |S| + k$. Since S' isaminimal opengeodetic dominating set of G',$u_i \notin S$ for all i, where $1 \leq i \leq k$. We showthat Sisa minimalopen geodetic dominating set of G. If $u_i \in I[S]$and $u_i \in N[S]$for all $u_i \in V(G) - S$, then S is a minimal open geodetic dominating set of G. Ifnot, then thereexists avertex $u_i \in V(G)$ suchthat $u_i \notin$ I[S] or$u_i \notin$N[S]. Thenthe set $S \cup \{u_i\}$ $(1 \leq i \leq k)$ is a minimal opengeodetic dominating set of G. Hence$\gamma_{og}^{+}(G') = |S| + k \leq \gamma_{og}^{+}(G) + k$.

Theorem 2.22. For any two integer a and n with $2 \leq a \leq n$, there exists a connectedgraph G with $\gamma_{og}^{+}(G) = a$ and $|V(G)| = n$.

Proof. It can be easily verified that the result is true for $2 \leq n \leq 3$. If $n = 2$, then$G = K_2$ and if$n = 3$, then G is either P_3 or K_3. For $n \geq 4$. If $a = n$, then G = K_n andif $a = n - 1$, then $G = K_{1,n-1}$. For $a \leq n - 2$. Let P: x, y, z be a path on three vertices.Let G be a graph obtainedfrom P by adding new vertices $z_1, z_2, ..., z_{a-3}, v_1, v_2, ..., v_{n-a}$and joining each $z_i (1 \leq i \leq a-3)$ with z, and joining each v_i $(1 \leq i \leq n-a)$ with $x$and z. Thegraph G is shown in Figure 6. Let S = {z_1, z_2, ..., z_{a-3}}. Then By Theorem 1.1 (i)S is a subset of every open geodeticdominating set. It is easily verified that $S \cup \{u\}$, and $S \cup \{u,v\}$ is not an open geodeticdominating set of G and so $\gamma_{og}^{+}(G) \geq a$. Now S'= S ∪ $\{x\}$ ∪ {y, v_i} $(1 \leq i \leq n-a)$or S'=S∪ $\{x\}$ ∪ $\{v_i, v_j\}$ $(1 \leq i,j \leq n-a)$ is a minimal open geodetic dominatingset of G and so γ_{og}^{+}(G)≥ a. We provethat $\gamma_{og}^{+}(G) = a$.If not, suppose that γ_{og}^{+}(G)$> a$. Then there exists a minimal open geodetic dominating set of $S''$with $|S''| \geq$ a + 1. Then S''contains atleast two$v_i$$(1 \leq i \leq n-a)$. Now $v_i$must lie on $I[x,z_j]$ for $(1 \leq i \leq n-a)$ and $(1 \leq j \leq a-3)$.Then x must belongs toS''Then it follows that $S' \subset S''$, which is a contradiction to $S''$is a minimalopen geodeticdominating set of G. Therefore $\gamma_{og}^{+}(G) = a$.



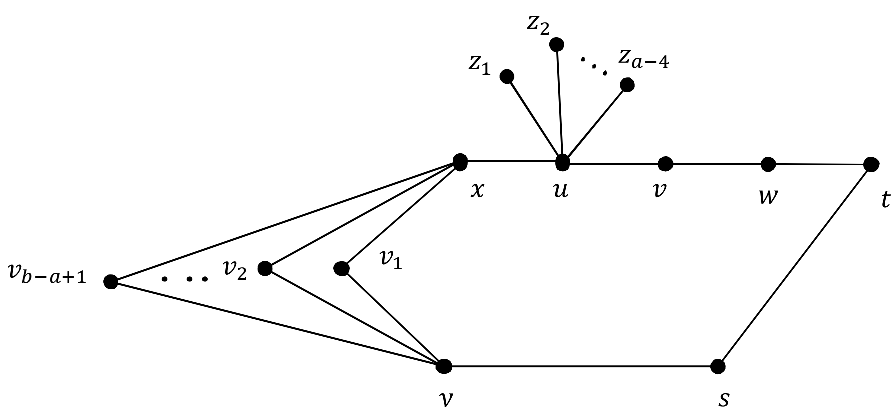
G
Figure 6

■

Theorem 2.23. For any two integer a and b with $2 \leq a \leq b$, there exists aconnected graph G with $\gamma_{og}(G) = a$, $\gamma_{og}^{+}(G) = b$.

Proof. It can be easily verified that the result is true for $2 = a = b$. Considerthe graph G =K_n. It is clear that $\gamma_{og}(K_2)$ = 2 , $\gamma_{og}^{+}(K_2)$ = 2. If $2 < a = b$, thenconsider the graph G = K_n (n > 2). It is clear that $\gamma_{og}(K_n)$ =$\gamma_{og}^{+}(K_n)$ = n. If $2 < a = b$, then consider the graph G = $K_{1,n}$. It is clear that $\gamma_{og}(K_{1,n})$ =$\gamma_{og}^{+}(K_{1,n})$ = $n-1$.Now we consider $2 < a < b$.Let $P\colon x, u, v, w, t$ be a path on five vertices. Let H be a graph obtained from Pbyadding new vertices z_1, z_2, ..., $z_{a-4}$and joining each z_i (1 ≤ i ≤ a − 4) with u. Let Gbe a graph obtained from H by adding new vertices y, s, v_1, v_2, ..., v_{b-a+1} and joiningeach $v_i$$(1 \leq i \leq b-a+1)$ with x and y and joint s with y and t, the graph G isshown in Figure 7.First we show that $\gamma_{og}(G) = a$. Let Z = {z_1, z_2, ..., z_{a-4}} be the set of all endvertices of G. By Theorem 1.1 (i)Z is a subset of every open geodetic dominating set of G. Itis easily verified that Z is not a open geodetic dominating set of G. It is easily verifiedthat $Z \cup \{x_1\}$ or $Z \cup \{x_1, x_2\}$ or $Z \cup \{x_1, x_2, x_3\}$ is not a open geodetic dominatingset where $x_1, x_2, x_3 \notin Z$ and so $\gamma_{og}(G) \geq a$. Now $S = Z \cup \{y, s, w, u\}$ is an opengeodetic dominating set of G so that $\gamma_{og}(G) = a$. Next we prove that $\gamma_{og}^{+}(G) = b$. Let$W = Z \cup \{v_1, v_2, \ldots, v_{b-a+1}, s, t, u\}$. Then W is an open geodetic dominating setof G and so $\gamma_{og}^{+}(G) \geq a - 4 + b - a + 1 + 3 = b$. First we prove that W is a minimal open geodetic dominating setof G. Suppose that W is not a minimal open geodetic dominating setof G. Then there exists $W' \subset$ W such that W' is a open geodetic dominating setof G. Hence there exists $z \in W$ such that $\notin W'$. By Theorem 1.1 (ii)$z \neq z_i(1 \leq i \leq a - 4)$. If $z = v_i(1 \leq i \leq b - a + 1)$ then W' is not a dominating setof G. If $z = s$ or t or u, then W' is not an open geodetic setof G. Hence W' is not an open geodetic dominating setof G. Therefore W is a minimalopen geodetic dominating setof G. Next we prove that γ_{og}^{+}(G) = b.Suppose that $\gamma_{og}^{+}(G) \geq b + 1$.Then there exists a open geodetic dominating set of Tsuch that $|T| \geq b + 1$. By Theorem 1.1(i)$Z \subset T$. Suppose that$v_i \notin$ T for some i. Then$s \in T$ and either v or $w \in T$. Let us assume that $v \in T$. Now s and$v$must lie onsome pair of vertices of T.

Which implies t must belongs to T. Hence Tcontains opengeodetic dominating set, which is a contradiction. Therefore $\gamma_{og}^{+}(G) = b$. ■



G

Figure 7

REFERENCES

1. Buckley.F and Harary.F, Distance in Graphs, *Addition- Wesley, Redwood City*, CA, 1990.
2. Buckley. F , Harary.F and Quintas.L.V., Extremal results on the geodetic number of a graph, Scientia1988;2A:17-26.
3. Caro. Y and Yuster. R, Dominating a family of graphs with small connected graphs, *Combin.Probab.Comput.*2000; 9: 309-313.
4. Chartrand .G, Harary.F and Zhang.P, Geodetic sets in graphs, *Discussiones MathematicaeGraph Theory*, 2000;20: 129-138.

5. Chartrand .G, Harary.F, Swart.H.C and Zhang.P. Geodomination in Graphs, *Bulletin of the ICA*, 2001; 31: 51 - 59.
6. Chartrand .G, Harary.Fand Zhang. P, On the geodetic number of a graph, *Network*s, 2002; 39(1): 1-6.
7. Cockayne.E. J, Goodman.S and Hedetniemi.S.T, A linear algorithm for the domination number of a tree, *Inform. Process.Lett*. 1975; 4: 41-44.
8. Douglas B. West, Introduction to graph theory, Second Edition, 2012.
9. Hansberg. A, Volkmann. L, On the Geodetic and Geodetic domination number of a graph, *Discrete Mat*hematics 2010; 310: 2140-2146.
10. Harary. F, Graph theory, *Addison-Wesley*, 1969.
11. Harary. F, Loukakis. E, Tsouros. C, The geodetic number of a graph. *Math..Comput. Modeling*1993;11: 89-95.
12. Robinson Chellathurai.S and Padma Vijaya.S, Geodetic domination in the corona and join of graphs,*Journal of Discrete Mathematical Sciences and Cryptography*, 2014; 17(1):81-90.
13. Robinson Chellathurai.Sand Vijimon Moni.V, The Open Geodetic domination number of a Graph, *International Journal of Pure and Applied Mathematics*, 2018;119(16): 5337-5348.
14. Santhakumaran. A.P and Kumari Latha.T, On the Open Geodetic Number of a Graph *Scientia, Series A : Mathematical Science*s, 2010;20: 131-142.