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### **PEBBLING $C_5 \times C_5$**

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#### **ABSTACT:**

Given a configuration of pebbles on the vertices of  $G$ , a pebbling move consists of taking two pebbles off some vertex  $v$  and putting one of them back on a vertex adjacent to  $v$ . A graph is called pebbleable if for each vertex  $v$  there is a sequence of pebbling moves that would place at least one pebble on  $v$ . The pebbling number  $f(G)$  of a graph  $G$  is the smallest integer  $m$  such that  $G$  is pebbleable for every configuration  $D$  of  $m$  pebbles on  $G$ . In this paper we find the pebbling number of  $C_5 \times C_5$ .

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**INTRODUCTION:**

Chung<sup>1</sup> conjectured that for any connected graph  $G$  and  $H$ ,  $f(G \times H) \leq f(G) \times f(H)$ . David S Herscovici and Aparna W Higgins<sup>2</sup>, proved the conjecture when  $G=H=C_5$ . Lourdusamy and Somasundaram<sup>3</sup>, proved the conjecture using linear programming. In both the references 2 and 3, they broke down the possible configuration into six steps and verified various situations. Using set pebbling we find the pebbling number of  $C_5 \times C_5$ . This method is better than the previous methods. In this method, instead of a single target vertex we are in addition using four vertices as target vertices. The structure of the proof (construction) would help us in proving, more generally,  $f(G_1 \times G_2 \times \dots \times G_n) = 5^n$ .

Lemma 1: *If there are at least 5 pebbles placed at the vertices of the graph  $G_1$  shown in Figure(1) below, at least one pebble can be moved to one of the vertex sets  $\{1,2\}$ ,  $\{3,4\}$ ,  $\{5,6\}$ , and  $\{7,8\}$ .*

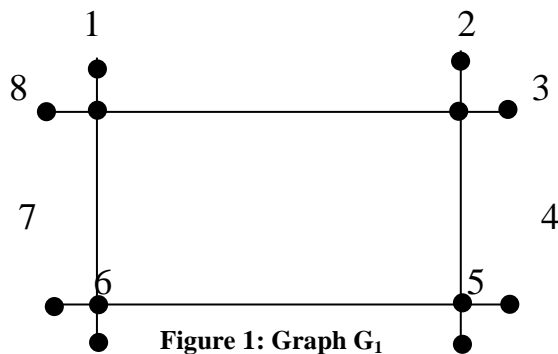


Figure 1: Graph  $G_1$

**Proof :** Let  $S = \{1,2\}$  be the target set. If a or b has 2 pebbles then the result is true. Otherwise, there are at least 3 pebbles remaining. They are to be placed on b or c. Now  $(a, b, c, d)$  has the distribution like  $(a, b, c, d) = (1,2,1,1)$ ,  $(0,3,2,0)$ , or  $(1,2,3,0)$  in some order. In all cases one pebble can be moved to  $\{1,2\}$ .

**Lemma 2:** From reference 3 we have,  $f(P_3 \times P_3) = f(P_3) \times f(P_3) = 2^2 \times 2^2 = 16$ .

We take the following notation from ref1.

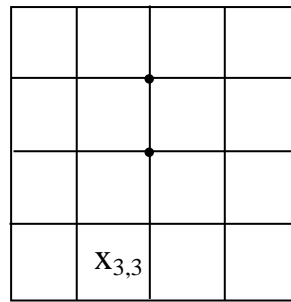


Figure 2. Representation of  $C_5 \times C_5$

Figure (2) represents the graph of  $C_5 \times C_5$  is another representation of the graph of  $C_5 \times C_5$ . Thus a 5 by 5 grid represents  $C_5 \times C_5$ . Let  $v_{ij}$ ,  $1 \leq i, j \leq 5$  represents the vertex set of  $C_5 \times C_5$ . Without loss of generality we assume  $v_{33}$  as the target vertex. Let  $P$  and  $Q$  represent cycles with vertices  $v_{i3}$ ,  $1 \leq i \leq 5$ , and  $v_{3j}$ ,  $1 \leq j \leq 5$  respectively and  $Q$  divides  $V(C_5 \times C_5) \setminus V(P \cup Q)$  into four parts. Choose one part as  $A$ , the parts which are symmetrically opposite to  $A$  with respect to  $P \cup Q$  as  $A^c$  and diagonally opposite to  $A$  as  $A^d$ . The situation is shown in figure (4).

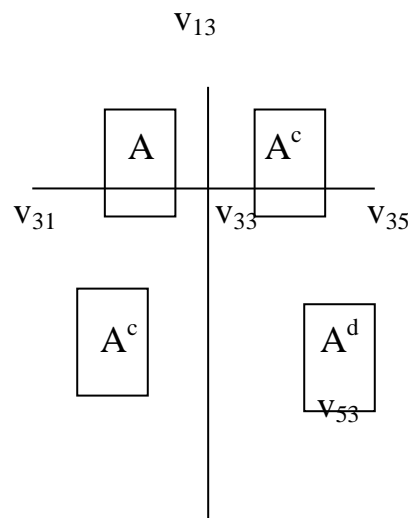


Figure 3 Representation of  $C_5 \times C_5$  as blocks

$$|V(A)| = |V(A^c)| = |V(A^d)| = 4.$$

Let  $V_{ij}$  represents the  $n^{\text{th}}$  vertex at a distance  $m$  from  $v_{33}$ . Let  $S = \{V_{ij}, 1 \leq i, j \leq 4\}$ , That is  $S = \{v_{23}, v_{32}, v_{34}, v_{43}\}$ . The pebbling number of  $C_5 \times C_5$  is the set pebbling number  $f(S, 2)$ . That is, by placing 2 pebbles at any vertex in  $S$ , one pebble can be moved to  $v_{33}$ . Let  $Z$  denotes the set of all vertices in  $P \cup Q$  at a distance 2 from  $v_{33}$ . That is,  $Z = \{v_{13}, v_{31}, v_{35}, v_{53}\}$ . For two vertices  $u$  and  $v$ , the distance  $d(u, v)$  is the length

of the shortest path from  $u$  to  $v$ . Let the cost of a pebble on a vertex  $u$  from a vertex  $v$  is  $2^{d(u,v)}$ . The cost of the pebbles on a path from  $u$  to  $v$ , from  $v$  is the total cost of all the pebbles on the path. Hence if the cost of the pebbles on any path with  $v_{33}$  as one of its end vertices from  $v$  is  $2^{d(v,v_{33})}$ , one pebble can be moved to  $v_{33}$ . Obviously, diameter of  $C_5=2$  and diameter of  $C_5 \times C_5 = 2 \times$  diameter of  $C_5=4$ . Hence if there are at least 16 pebbles on any one of the vertices of  $C_5 \times C_5$  at least one pebble can be moved to any desired vertex in  $C_5 \times C_5$ .

We prove that  $f(C_5 \times C_5) \leq 25$ , by considering various pebbling situation .

Proof :

Notation : By  $A \rightarrow A^c$  we mean, by removing  $y$  pebbles from  $A$ ,  $x$  pebbles can be moved to  $A^c$ .

$|Z|$  denotes the number of pebbles in  $Z$  and  $|v|$  denote the number of pebbles at  $v$ .

Assumption: If  $P$  is a path with  $v_{33}$  as one end vertex, without loss of generality, we may assume that the total cost of all the pebbles in the path must be less than 16 with respect to the other end vertex.

(A): Suppose  $|Z|=|S|=0$

**Case 1:** Let  $14 \leq |A| \leq 15$ .

By assumption, in the worst situation all pebbles must be at  $v_{11}$ .

(a) If  $|A^c| \geq 2$ , then  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{8} A^c, A^c \xrightarrow{5} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(b) If  $|A^c| \leq 1$ , then  $|A^d| \geq 8$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (2,1,0,5), (2,0,0,6), (1,1,1,5), (1,0,0,7).$$

Now  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{6} A^c, v_{55} \xrightarrow{4} A^c, A^c \xrightarrow{5} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

**Case 2:** Let  $12 \leq |A| \leq 13$ . By assumption, in the worst situation all pebbles must be at  $v_{11}$ .

(a) If  $|A^c| \geq 3$ , then  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{4} A^c, A^c \xrightarrow{5} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(b) If  $1 \leq |A^c| \leq 2$ , then  $|A^d| \geq 8$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (2,0,1,5), (2,0,0,6), (1,1,1,5), (1,0,0,7).$$

Now  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{4} A^c, v_{55} \xrightarrow{4} A^c, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(c) If  $|A^c| = 0$ , then  $12 \leq |A^d| \leq 13$ .

Let  $|A^d|=13$  and  $|A|=12$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$(v_{44}, v_{45}, v_{54}, v_{55}) = (1,0,0,11), (1,1,0,10), (0,0,0,12)$ .

Now  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{4} A^c, v_{55} \xrightarrow{10} A^c, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$

**Case 3:** Let  $10 \leq |A| \leq 11$ . By assumption, in the worst situation all pebbles must be at

$v_{11}$

(a) If  $|A^c| \geq 4$ , then  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{2} A^c, A^c \xrightarrow{5} v_{32}, v_{32} \xrightarrow{2} v_{33}$

(b) If  $|A^c| = 3$ , then  $|A^d| \geq 8$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$(v_{44}, v_{45}, v_{54}, v_{55}) = (2,0,1,5), (2,0,0,6), (1,1,1,5), (1,0,0,7)$

Now  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{2} A^c, v_{55} \xrightarrow{2} A^c, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(c) If  $1 \leq |A^c| \leq 2$ , then  $|A^d| \geq 10$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration

$(v_{44}, v_{45}, v_{54}, v_{55}) = (1,0,0,9), (1,1,0,8), (1,1,1,7), (0,1,1,8), (0,1,0,9), (0,0,0,10)$

Now  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{2} A^c, v_{55} \xrightarrow{3} A^c, A^c \xrightarrow{5} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(d) If  $|A^c| = 0$ , then  $|A^d| = 14$ . Interchange  $A$  and  $|A^d|$  and follow pebbling steps accordingly.

**Case 4 :** Let  $8 \leq |A| \leq 9$ . By assumption, in the worst situation all pebbles must be at

$v_{11}$

(a) If  $|A^c| \geq 5$ , then  $v_{11} \xrightarrow{8} v_{32}, v_{11} \xrightarrow{4} A^c, A^c \xrightarrow{8} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(b) If  $3 \leq |A| \leq 4$ , then  $|A^d| \geq 8$ . Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$(v_{44}, v_{45}, v_{54}, v_{55}) = (2,0,1,5), (2,0,0,6), (1,1,1,5), (1,0,0,7)$ .

Now  $v_{11} \xrightarrow{8} v_{32}, v_{55} \xrightarrow{2} A^c, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(c) If  $|A^c| \leq 2$ , then  $|A^d| \leq 12$ . Interchange  $|A|$  and  $|A^d|$

Case 5: Let  $0 \leq |A| \leq 7$ . Interchange  $|A|$  and  $|A^d|$

**(B): Suppose  $|Z| \leq 4, |S| = 0$**

Let  $|v_{31}| = 1$ . If  $|A| \geq 12$ , and by assumption that in the worst situation all pebbles must be at  $v_{11}$ , three pebbles can be moved to  $v_{13}$ . Now  $v_{13} \rightarrow v_{33}$ . So we need to consider  $0 \leq |A| \leq 11$  only.

**Case (1): Let  $9 \leq |A| \leq 11$ .**

- A. If  $|A^c| \geq 5$ , then  $v_{11} \xrightarrow{\frac{1}{8}} v_{32}, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$ .
- B. If  $|A^c| = 4$ , then  $|A^d| \geq 2$
- C. Now,  $v_{11} \xrightarrow{\frac{1}{4}} v_{31}, v_{31} \xrightarrow{\frac{1}{2}} v_{32}, v_{11} \xrightarrow{\frac{1}{2}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$ .
- D. If  $|A^c| = 3$ , then  $|A^d| \geq 4$
- E. Now,  $v_{11} \xrightarrow{\frac{1}{4}} v_{31}, v_{31} \xrightarrow{\frac{1}{2}} v_{32}, v_{11} \xrightarrow{\frac{2}{4}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$ .
- F. If  $|A^c| = 2$ , then  $|A^d| \geq 6$ .
- G. Now,  $v_{11} \xrightarrow{\frac{1}{4}} v_{31}, v_{31} \xrightarrow{\frac{1}{2}} v_{32}, v_{11} \xrightarrow{\frac{2}{4}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$ . If  $|A^c| = 1$ , then  $|A^d| \geq 8$

Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (1,1,1,5), (1,1,0,6), (1,0,0,7), (0,0,0,8), (0,1,0,7), (0,1,1,6).$$

Now,  $v_{11} \rightarrow v_{31}, v_{31} \rightarrow v_{32}, v_{11} \xrightarrow{\frac{1}{2}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$

(a) If,  $|A^c| = 0$ , then  $|A^d| \geq 10$ .

Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55})$$

$$v_{11} \rightarrow v_{31}, v_{31} \xrightarrow{\frac{1}{4}} v_{32}, v_{11} \xrightarrow{\frac{1}{2}} A^c, v_{55} \xrightarrow{\frac{2}{4}} A^c, A^c \xrightarrow{\frac{3}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$$

**Case (2): Let  $6 \leq |A| \leq 8$ .**

(a) If  $|A^c| \geq 4$ , then  $|A^d| \leq 5$

Now  $v_{11} \rightarrow v_{31}, v_{31} \xrightarrow{\frac{1}{4}} v_{32}, v_{11} \xrightarrow{\frac{1}{2}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$

(b) If  $|A^c| = 3$ , then  $|A^d| \geq 7$

Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (2,0,0,5), (2,0,1,4), (2,1,1,3), (1,1,1,4), (1,1,0,5), (1,0,0,6), (0,0,0,7), (0,1,0,6), (0,1,1,5)$$

Now,  $v_{11} \xrightarrow{\frac{1}{4}} v_{31}, v_{31} \xrightarrow{\frac{1}{2}} v_{32}, v_{55} \xrightarrow{\frac{1}{2}} A^c, v_{11} \xrightarrow{\frac{1}{2}} A^c, A^c \xrightarrow{\frac{1}{5}} v_{32}, v_{32} \xrightarrow{\frac{1}{2}} v_{33}$ .

(c) If  $1 \leq |A^c| \leq 2$ , then  $|A^d| \geq 9$

Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (1,0,0,8), (1,1,0,7), (1,1,1,6), (0,1,1,7), (0,1,0,8), (0,0,0,9)$$

$$\text{Now, } v_{11} \xrightarrow{1} v_{31}, v_{31} \xrightarrow{1} v_{32}, v_{11} \xrightarrow{1} A^c, v_{55} \xrightarrow{1} A^c, A_6^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{1} v_{33}. \quad \begin{matrix} & & & & & 2 \\ & & & & & 1 \end{matrix}$$

(d) If  $|A^c| = 2$ , then  $|A^d| \geq 6$ .

**Case (3):** Let  $4 \leq |A| \leq 5$ .

(a) If  $|A^c| \geq 5$ , then  $v_{11} \xrightarrow{4} v_{31}, v_{32} \xrightarrow{2} v_{33}, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

(b) If  $|A^c| \leq 2$ , then  $|A^d| \geq 8$

Here  $(v_{44}, v_{45}, v_{54}, v_{55})$  has any one of the following configuration.

$$(v_{44}, v_{45}, v_{54}, v_{55}) = (1,0,0,8), (1,1,0,7), (1,1,1,6), (0,1,1,7), (0,1,0,8), (0,0,0,9)$$

$$\text{Now, } v_{11} \xrightarrow{4} v_{31}, v_{31} \xrightarrow{2} v_{32}, v_{55} \xrightarrow{4} A^c, A^c \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}.$$

(c) If  $|A^c| \leq 2$ , then  $|A^d| \geq 12$  .. Interchange  $|A|$  and  $|A^d|$

Case (4): Let  $0 \leq |A| \leq 3$ , Choose  $A$  or  $A^d$  with more number of pebbles and do the pebbling steps accordingly.

**(C) Suppose  $|Z| = 0, |S| \leq 4$**

Let  $|v_{32}| = 1$ .

(a) If  $|A| \geq 5$ , then  $A \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ . So we assume that  $|A| \leq 4$

(b) If  $|A^c| \geq 5$ , then  $A^c \xrightarrow{1} v_{32}, v_{32} \xrightarrow{1} v_{33}$

(c) If  $|A^c| \leq 4$ , then  $|A^d| \geq 13$ . Interchange  $|A|$  and  $|A^d|$

**(D) Suppose  $|Z| \leq 4, |S| \leq 4$**

**Case(1)** Let  $|A|, |A^c|$  or  $|A^d| \geq 5$ . Suppose  $|A| \geq 5$ . Now  $A \xrightarrow{3} v_{32}, v_{32} \xrightarrow{2} v_{33}$ .

Case(2) Let  $|A| \leq 4$ . If  $|A^c| \leq 4$ , then  $|A^d| \geq 9$ . Interchange  $|A|$  and  $|A^d|$

We proved that  $f(C_5 \times C_5) \leq 25$

By placing one pebble each on the vertices of  $(C_5 \times C_5)$  except the target vertex, no pebble can be moved on to the target vertex. That is  $f(C_5 \times C_5) \geq 25$  and hence,  $f(C_5 \times C_5) = 25$ .

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