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The Forcing Monophonic Hull Domination Number of a Graph

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ABSTRACT

For a connected graph G = (V, E), a monophonic hull set M in a connected graph G is called a monophonic hull dominating set of G if M is both monophonic hull set and a dominating set of G. The monophonic hull domination number rmh(G) of G is the minimum cardinality of a monophonic hull dominating set of G .Let M be a minimum monophonic hull dominating set of G. A subset T $\subseteq$ M is called a forcing subset for M if M is the unique minimum monophonic hull dominating set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull domination number of M, denoted by $f_{rmh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull domination number of G, denoted by $f_{rmh}(G)$, is $f_{rmh}mh(G) = min\{f_{rmh}(M)\}$, where the minimum is taken over all minimum monophonic hull dominating sets M in G. Some general properties satisfied by this concept is studied. The monophonic hull domination number of certain standard graphs are determined.. The forcing monophonic hull domination number of a connected graph to be 0 and 1 are characterized. It is shown that for every positive integers a and b with 0≤ a < b and b ≥ 2, b > a+1, there exists a connected graph G with $r_{mh}(G)$ = a and $f_{rmh}(G)$= b.

KEYWORDS: domination number, monophonic hull number, monophonic hull domination number, forcing monophonic hull domination number.

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INTRODUCTION

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. A convexity on a finite set V is a family C of subsets of V, convex sets which are closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected sub graph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W . The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u–v$ path in G. An $u–v$ path of length $d(u,v)$ is called an $u–v$ geodesic. A *chord* of a path $u_o, u_{1,} . . . ., u_n$ is an edge u_iu_j with $j \geq i + 2$. $(0 \leq i, j \leq n)$. A u - v path P is called *monophonic* if it is a chordless path. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \cup_{u,v \in S} I[u, v]$. The set S is *convex* if $I[S] = S$. Clearly if $S = \{v\}$or $S = V$, then S is convex. The *convexity* number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the *convex hull* of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A The *hull number* $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a *minimum hull set* or simply a *h- set* of G. A vertex x is said to lie on a u-v monophonic path P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $J [u, v]$ denotes the set of all vertices which lie on u - v monophonic path. For a set S of vertices, let $J_m[S] = U_{u,v \in S} J[u,v]$. The set S is *monophonic convex* or *m-convex* if $J_m [S] = S$. The monophonic convexity number, denoted by $C_m(G)$ is the cardinality of a maximum proper monophonic convex subset of V.The smallest monophonic convex set containing S is denoted by $J_{mh}(S)$ and called the monophonic convex hull of S. Since the intersection of two m-convex sets is m-convex, the monophonic hull is well defined. Note that $S \subseteq J_m(S) \subseteq J_{mh}(S) \subseteq V$. A subset $S \subseteq V$ is called a monophonic set if $J_m(S) = V$ and a monophonic hull set if $J_{mh}(S)=V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a m-set of G. The monophonic hull number $mh(G)$ of G is the minimum order of its monophonic hull sets and any monophonic hull set of order $mh(G)$ is a minimum monophonic hull set or simply a mh-set of G. A set of vertices D in a graph G is a dominating set if each vertex of G is dominated by some vertex of D. The domination number of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a γ-set. A monophonic hull set M in a connected graph G is called a *monophonic hull dominating set* of G if M is both monophonic hull set and a dominating set of G. *The monophonic hull domination number* $\gamma_{mh}(G)$ of G is the minimum cardinality of a monophonic hull dominating set of G. Any monophonic hull dominating set of cardinality $\gamma_{mh}(G)$ is called γ_{mh}-set of G. For the graph G given in Figure 1.1, $S = \{v_1, v_8\}$ is a *mh*- set of G so that $mh(G) = 2$ and also $S_1 = \{v_2, v_4, v_7\}$ and $S_2 = \{v_2, v_5, v_7\}$are a γ -sets of G so that $\gamma(G) = 3$. Also $M_1 = \{ v_1, v_3, v_6, v_7\}$ is a γ_{mh} -set of G so that $\gamma_{mh}(G) = 4$. Two vertices u and v are said to be independent if they are not adjacent. A vertex v is an *extreme vertex* of a graph G if the sub graph induced by its neighbors is complete. Throughout the following G denotes a connected graph with at least two vertices.

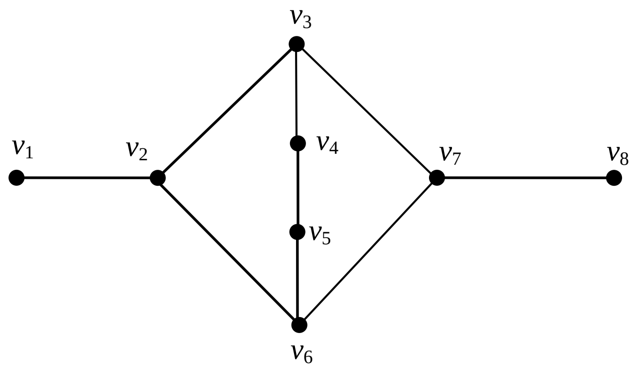

Figure 1.1

A recent application of the convex hull is the Onion Technique for linear optimization queries. This method is based on a theorem that a point which maximizes an arbitrary multidimensional weightening function can be found on the convex hull of the data set. The skyline of a dataset can be used to determine various point of a data sets which could optimize an unknown objective in the user's intentions. E.g. users of a booking system may search for hotels which are cheap and close to the beach. The skyline of such a query contains all possible results regardless how the user weights his criteria *beach* and *cost*. The skyline can be determined in a very similar way as the convex hull.

\The following theorem is used in the sequel.

Theorem 1.1.[17] *Each extreme vertex of a connected graph G belongs to every monophonic hull dominating set of G.*

2. The forcing monophonic hull domination number of a graph

Definition 2.1. Let G be a connected graph and M a minimum monophonic hull dominating set of G. A subset $T \subseteq M$ is called a *forcing subset* for M if M is the unique minimum monophonic hull dominating set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of* M. The *forcing monophonic hull domination number* of M, denoted by $f_{rmh}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing monophonic hull domination number* of G, denoted by $f_{rmh}(G)$, is $f_{rmh}mh(G) = min\{f_{rmh}(M)\}$, where the minimum is taken over all minimum monophonic hull dominating sets M in G.

Example 2.2. For Consider the graph G given in Figure 2.1. The sets $M_1 = \{v_1, v_3\}$, $M_2 = \{v_1, v_6\}$ and $M_3 = \{v_3, v_5\}$ are the only three rmh-sets of G such that $f_{rmh}(M_1) = 2, f_{rmh}(M_2) = 1$ and $f_{rmh}(M_3)$ so that $f_{rmh}(G) = 1$.

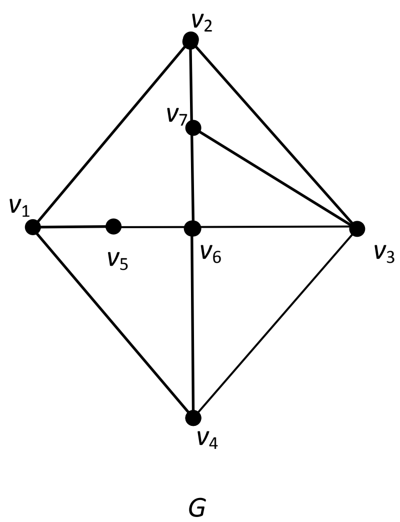

Figure 2.1

The next theorem follows immediately from the definitions of the monophonic hull domination number of a connected graph G.

Theorem 2.3. For every connected graph $G, 0 \leq f_{rmh}(G) \leq rmh(G)$.

Definition 2.4. A vertex v of a graph G is said to be a *monophonic hull dominating vertex* if v belongs to every rmh-set of G.

The following theorems characterizes graphs for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{rmh}(G) = 1$.

Theorem 2.5. Let G be a connected graph. Then

a) $f_{rmh}(G) = 0$ if and only if G has a unique rmh −set.

b) $f_{rmh}(G) = 1$ if and only if G has at least two rmh-sets, one of which is a unique rmh-set containing one of its elements, and

c) $f_{rmh}(G) = rmh(G)$ if and only if no mh-set of G is the unique rmh-set containing any of its proper subsets.

Proof. (a) Let $f_{rmh}(G) = 0$. Then by definition, $f_{rmh}(M) = 0$ for some rmh-set S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique rmh-set of G. The converse is clear.

(b) Let Let $f_{rmh}(G) = 1$. Then by Theorem 2.5 (a), G has at least two rmh-sets. Also, since $f_{rmh}(G) = 1$, there is a singleton subset T of a rmh-set S of G such that T is not a subset of any other rmh-set of G. Thus S is the unique rmh-set containing one of its elements. The converse is clear.

(c) Let $f_{rmh}(G) = rmh(G)$. Then $f_{rmh}(M) = rmh(G)$ for every rmh-set M in G. Also, by Theorem 2.3, $rmh(G) \geq 2$ and hence $f_{rmh}(G) \geq 2$. Then by Theorem 2.5 (a), G has at least two rmh-sets and so the empty set ϕ is not a forcing subset for any rmh-set of G. Since $f_{rmh}(M) = rmh(G)$, no proper subset of M is a forcing subset of M. Thus no rmh-set of G is the unique rmh-set containing any of its proper subsets. Conversely, the data implies that G contains more than one rmh-set and no subset of any rmh-set S other than S is a forcing subset for M. Hence it follows that $f_{rmh}(G) = rmh(G)$. ■

Theorem 2.6. Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective rmh-sets in G. Then $\bigcap_{F \in \Im} F$ is the set of monophonic hull dominating vertices of G.

Proof. Let W be the set of all connected hull vertices of G. We are to show that $W = \bigcap_{F \in \Im} F$. Let $v \in W$. Then v is a monophonic hull dominating vertex of G that belongs to every rmh-set S of G. Let $T \subseteq S$ be any minimum forcing subset for any rmh-set S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique rmh-set containing T' so that T' is a forcing subset for S with $|T'| < |T|$ which is a contradiction to T is a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F \in \Im} F$ so that $W \subseteq \bigcap_{F \in \Im} F$.

Conversely, let $v \in \bigcap_{F\in\mathfrak{F}} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, we have $F \subseteq S$ and thus $v \in S$ for every S, which implies that v is a monophonic hull dominating vertex of G. Thus $v \in W$ and so $\bigcap_{F\in\mathfrak{F}} F \subseteq W$. Hence $W = \bigcap_{F\in\mathfrak{F}} F$

Corollary 2.7. Let G be a connected graph and S a rmh −set of G. Then no monophonic hull dominating vertex of G belongs to any minimum forcing subset of S.

Theorem 2.8. Let G be a connected graph and S be the set of all monophonic hull dominating vertices of G. Then $f_{rmh}(G) \leq rmh(G) - |S|$.

Proof. Let M be any rmh-set of G. Then $rmh(G) = |M|$, $W \subseteq M$ and W is the unique rmh-set containing $M - W$. Thus $f_{hc}(G) \leq |M - W| = |M| - |W| = rmh(G) - |W|$. ■

Corollary 2.9. If G is a connected graph with k extreme vertices, then $f_{rmh}(G) \leq rmh(G) - k$.

Proof. This follows from Theorem 1.1 and Theorem 2.8. ■

Theorem 2.10. For any complete graph $G = K_p$ $(p \geq 2)$ or any non-trivial tree $G = T$, $f_{rmh}(G) = 0$.

Proof. For the complete graph $G = K_p$, it follows from Theorem 1.1 that the set of all vertices of G is the unique monophonic hull dominating set of G. Hence it follows from Theorem 2.5(a) that $f_{rmh}(G) = 0$. For any non-trivial tree G, the monophonic hull domination number $rmh(G)$ equals the number of end vertices in G. In fact, the set of all end vertices of G is the unique rmh −set of G and so $f_{rmh}(G) = 0$ by Theorem 2.5(a). ■

Theorem2.11. For a complete bipartite graph $G = K_{r,s}$

$$f_{rmh}(G) = \begin{cases} 0 \; ; r = 1, \quad s \geq 2 \\ 1 \; ; r = 2, s \geq 2 \\ 2 \; ; 3 \leq r \leq s \end{cases}$$

Proof. If $r = 1$, $s \geq 2$, the result follows from Theorem 2.10. For $r = 2, s \geq 2$, let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, ..., v_s\}$ be a bipartition of G. Then $M = \{u_1, u_2\}$ is a mh-set of G. It is clear that M is the only rmh-set containing u_1 so that $f_{rmh}(G) = 1$. For $3 \leq r \leq s$, let $U = \{u_1, u_2, ..., u_r\}$ and $V = \{v_1, v_2, ..., v_s\}$ be a bipartition of G. Let $M_1 = \{u, v\}$. Suppose that u and v are adjacent. Then uv is a chord for the path $u - v$ and so $\{u, v\}$ is not a monophonic hull dominating set of G. Therefore u and v are independent. It is clear that M_1 is a monophonic hull dominating set of G. so that $mh(G) = 2$ and by Theorem 3.3, $0 \leq f_{rmh}(G) \leq 2$. Suppose $0 \leq f_{rmh}(G) \leq 1$. Since $rmh(G) = 2$ and the rmh –set of G is not unique, by Theorem 2.5 (b), $fmh(G) = 1$. Let $S = \{u, v\}$ be a rmh-set of G. Let us assume that $f_{rmh}(G) = 1$. By Theorem 2.5(b), S is the only rmh-set containing u or v. Let us assume that S is the only mh-set containing u. Then $r = 2$, which is a contradiction to $r \geq 2$. Therefore $f_{rmh}(G) = 2$. ■

In view of Theorem 2.3, we have the following realization result.

Theorem 2.12. For every pair a, b of integers with $0 \leq a \leq b$,$b \geq 2$ and there exists a connected graph G such that $f_{rmh}(G) = a$ and $rmh(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 1.1, $rmh(G) = b$ and by Theorem 2.10, $f_{rmh}(G) = 0$. For $a \geq 1$, Let $C_i: u_i, v_i, w_i, x_i, y_i$; $(1 \leq i \leq a)$ be a copy of the cycle C_5. Let D_i be the graph obtained from C_i by joining the vertex v_i with x_i and $y_i (1 \leq i \leq a)$. Let G be the graph obtained from $D_i (1 \leq i \leq a)$ by adding new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$. and joining x with each $u_{i,}$, $w_{i,}$ $(1 \leq i \leq a)$ and joining x with each $w_{i,}$ $(1 \leq i \leq b - a - 1)$.

The graph G is shown in Figure 2.2.. Let $Z = \{z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end-vertices of G. By Theorem 1.1, Z is a subset of every monophonic hull dominating set of G. For $1 \leq i \leq a$, let $F_i = \{x_i, y_i\}$. We observe that every rmh –set of G must contain at least one vertex from each F_i. Also it is easily observed that every rmh contains $\{x\}$ and so that $rmh(G) \geq b - a + a = b$. Now $M_1 = Z \cup \{x, x_1, x_2, x_3, \ldots x_a\}$ is a monophonic hull dominating set of G so that $rmh(G) \leq b - a + a = b$. Thus $rmh(G) = b$. Next we show that $f_{rmh}(G) = a$. Since every rmh-set contains $Z \cup \{x\}$, it follows from Theorem 2.8 that $f_{rmh}(G) \leq mh(G) - |Z \cup \{x\}| = b - (b - a) = a$. Now, since $rmh(G) = b$ and every rmh –set of G contains $Z \cup \{x\}$, it is easily seen that every rmh-set M is of the form $Z \cup \{x\} \cup \{d_1, d_2, d_3, \ldots d_a\}$, where $d_i \in F_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_{rmh}(G) = a$. ■

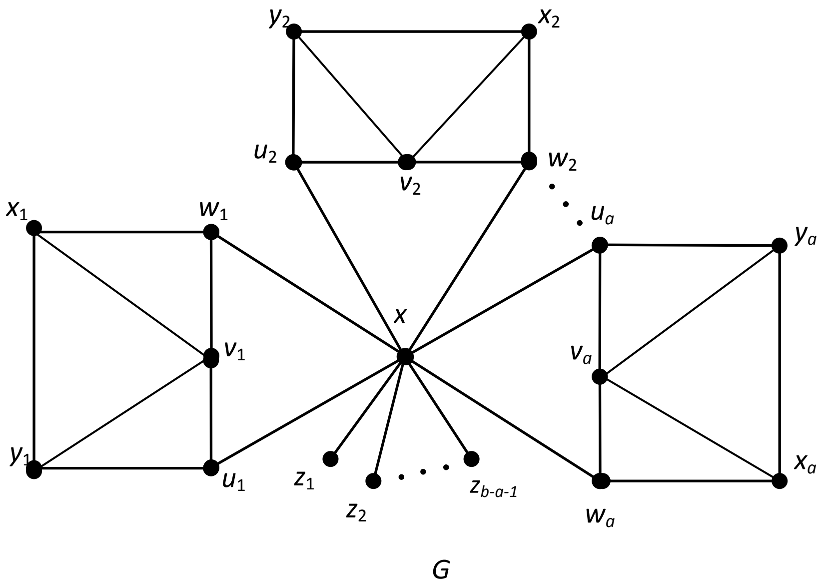

Figure 2.2

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