Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

Local Behavor Of The Discrete Quadratic Spline Interpolator

Dubey Suyash [1*]. Dubey P. [2] and S.S.Rana[3]

[1]Gyan Ganga Institute of Technology & Sciences, Jabalpur. M.P.India
[2]Swami Vevakanad University Sager M.P.India
[3]Rani Durgavaty University Jabalpur M.P. India

ABSTRACAT

In this paper we have defined discrete quadratic spline and estimated a precise error estimate concerning discrete quadratic spline interpolant matching the given functional values at mid point between the successive mesh points.

KEYWORLDS AND PHRASES: Discrete quadratic spline, interpolation, mid point interpolation, precise error estimate..

Mathematics Subject Classification 2010. 41A05, 41A15, 65D07

***Corresponding author**

Suyash Dubey

Department of Mathematics

Gyanganga College of Technology And Science Jabalpur

M.P. India 482003

Mail Id- suyash.dubey24@gmail.com Mob no 918770171940

INTRODUCTION

Let us consider a mesh P of [0, 1] given by $0 = x_0 < x_1 < ... < x_n = 1$ such that $x_i - x_{i-1} = p = \frac{1}{n}$ for $i = 1,2,....,n$. For a given h > 0 suppose a real continuous function $s(x,h)$ defined over [0,1] and its restriction to x_{i-1}, x_i is a polynomial s_i of degree 2 or less for i = 1,2,...., n. Then $s(x, h)$ defines a discrete quadratic spline if

$$D_h^{\{1\}} s_i(x_i - h) = D_h^{\{1\}} s_{i+1}(x_i + h), \; i = 1,2,.....,n-1 \qquad (1.1)$$

where the central difference operator $D_h^{\{1\}} f(x) = (f(x+h) - f(x-h))/2h$ (see Rana [10]). D(2,P,h) denotes the class of all discrete quadratic splines which satisfies the periodic condition.

Discrete splines have been introduced by [9] in connection with certain studies of minimization problems involving differences. Existence, uniqueness and convergence properties of discrete cubic spline interpolant matching the given function values at mesh point have been studied by[8] . For this case further studies in the direction of the result proved in [8] have been made by [3,4, 5,6 ,7, 11, 14] . Now [12] have obtained a precise estimate concerning the deficient discrete cubic spline interpolant matching the given function at two intermediate points between the successive mesh points. [12] observed that the local behavior of the derivative of a cubic spline interpolator is some times used to smooth a histogram which has been estimated by [13]. For application of discrete splines to solve general type of vibrational problem we refer to [15]. It may be observed that the approach used by [8] for defining discrete cubic splines is not capable of providing the corresponding definition for discrete quadratic spline and study its local behavior interpolating the given function at mid points.

EXISTENCE AND UNIQUENESS.

Considering the interpolatory condition for a given function f

$$s(t_i, h) = f(t_i), t_i = (x_i + x_{i-1})/2, \quad i = 1,2,....,n \qquad (2.1)$$

we shall prove the following :

THEOREM 2.1. Let f be 1 periodic and $p \geq 4h$, them for any h>0 there exists a unique 1 periodic discrete quadratic spline $s(x,h)$ in the class $D(2,P,h)$ which satisfies the interpolatory condition (2.1).

Proof. Suppose in the interval $[x_{i-1}, x_i]$ for all i,

$$2(p-2h)x(x,h) = (x - x_{i-1} - h)^2 M_i - (x_i - x - h)^2 M_{i-1} + 2(p-2h)c_i, \qquad (2.2)$$

where $M_i = M_i(h) = D_h^{\{1\}} s(x_i - h, h)$ and c_i is a constant which has to be determined.

We get the following from (2.2) when we appeal the interpolatory condition (2.1).

$$8f(t_i)=(p-2h)(M_i-M_{i-1})+8c_i \tag{2.3}$$

Since $s(r,h)$ is continuous, therefore using the continuity of $s(x,h)$ in (2.2) along with (2.3), we have

$$((p-4h)/2)M_{i-1}+(3p-4h)M_i+((p-4h)/2)M_{i+1}=F_i(h),\ i=1,2,....,n-1 \tag{2.4}$$

Where $F_i(h)=4(p-2h)(f(t_{i+1}\_-f(t_i)))/p$.

It may be seen that the excess of the positive value of the coefficient of M_i over the sum of the positive values of the coefficients of M_{i-1} and M_{i+1} is $2p$ which is >0. Thus, the coefficient matrix of the system of equations (2.4) is diagonally dominant and hence invertible. This completes the proof of Theorem 2.1.

ESTIMATION OF THE INVERSE OF THE COFFICIENT MATRIX .

Ahlberg, Nilson and Walsh[1] have estimated precisely the inverse of the coefficient matrix appearing in the studies concerning continuous cubic splines matching the given function at the mesh points. Following Ahlberg, Nilson and Walsh, we shall obtain similar precise estimate for the inverse of the coefficient matrix (2.4). It may be mentioned that this method permits the immediate application to the spline to standard problem of numerical analysis (see ANW[1], p.34). Without loss of generality we assume for the rest of the paper that discrete quadratic spline $s(x,h)$ under consideration satisfies the condition $D_h^{\{1\}}s(x_0-h,h)=0$. Now in order to find the inverse of the coefficient matrix of (2.4), we introduce the following square matrix of order *n* as

$$D_n(\alpha,\beta)=\begin{bmatrix} 2\beta & \alpha & 0 & . & . & . & 0 & 0 & 0 \\ \alpha & 2\beta & \alpha & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \alpha & 2\beta & \alpha \\ 0 & 0 & 0 & . & . & . & 0 & \alpha & 2\beta \end{bmatrix}$$

where α and β are given real numbers such that $\beta^2 \geq \alpha a^2$. By using the induction hypothesis it is easily seen that the determinant $|D_n|$ satisfies the following difference equation.

$$|D_n(\alpha,\beta)|-2\beta|D_{n-1}(\alpha,\beta)|+\alpha^2|D_{n-2}(\alpha,\beta)|=0 \tag{3.1}$$

with $|D_{-1}(\alpha,\beta)|=0, |D_0(\alpha,\beta)|=1$ and for $\eta=(\beta^2-\alpha^2)^{1/2}$

$$2\eta|D_n(\alpha,\beta)|=(\beta+\eta)^{n+1}-(\beta-\eta)^{n+1},\ \beta^2>\alpha^2$$

$$|D_n(\alpha,\beta)| = (n+1)\beta^n, \text{ otherwise.} \qquad (3.2)$$

Now we write the system of equations (2.4) in the form

$$AM = F \qquad (3.3)$$

Where the coefficient matrix A is a square matrix of order n-1 and M and F are the column vectors $[M_1, M_2, ..., M_{n-1}]$ and $[F_1, F_2, ..., F_{n-1}]$ respectively. Further in view of (3.1) and (3.2) it may be observed that

$$2q^{-n}(\beta+\alpha^2 2)|D_n(\alpha,\beta)| = 2\beta(1-(ar)^{2n} + \alpha^2 r(1-(ar)^{2n-2}) \qquad (3.4)$$

where $r = -q^{-1} = -(\beta - (\beta^2 - \alpha^2)^{1/2})/\alpha^2$.

Taking $2\beta = 3p - 4h$ and $\alpha = (p-4h)/2$ in $|D_n(\alpha,\beta)|$ we see from (3.1) that the determinant of the coefficient matrix A of (3.3) satisfies the difference equation.

$$|A| = 2\beta|D_{n-2}(\alpha,\beta)| - \alpha^2|D_{n-3}(\alpha,\beta)| \qquad (3.5)$$

Thus, it follows from (3.4) that

$$2q^{2-n}(\beta+\alpha^2 r)|A| = (2\beta+\alpha^2 r^2)^2 - \alpha^2(1+2\beta r)^2(\alpha r)^{2_{n-3}} \qquad (3.6)$$

Thus, substituting $2\beta = 3p-4h$ and $\alpha = (p-4h)/2$ in (3.5), we write the elements a_{ij} of A^{-1} from the cofactors of the transpose matrix (see [1], p. 35-38). Thus, for $0 < i \le j \le n-2$ or $i=j=0$

$$|A|a_{ij} = (q.r)^{j-i} D_i\left(\frac{p-4h}{2}, \frac{3p-4h}{2}\right) D_{n-j}\left(\frac{p-4h}{2}, \frac{3p-4h}{2}\right)$$

and

$$|A|a_{0j} = (q.r)^j D_{n-j}\left(\frac{p-4h}{2}, \frac{3p-4h}{2}\right) \text{ for } 0 < j \le n.$$

Now using (3.5) and (3.6) we see that for $0 < i \le j \le n-2$

$$((3p-4h) + r(1-3^{2n})a_{ij} = r^{j-i}(1-r^{2i+2})(1-2^{2(n-1-j)}),$$

$$((3p-4h) + r/2(1-r^{2n})a_{in-2} = r^{n-i-2}(1-r^{2i+2}), \text{ for } 0 < i \le n-2,$$

$$((3p-4h) + r/2)(1-r^{2n})a_{0j} = r^j(1-r^{2(n-j-1)}), \text{ for } 0 < j < n-2,$$

$$((3p-4h) + r/2)^2(1-r^{2n})a_{0n-2} = r^{n-2}(3p-4h+r).$$

From the above expressions, we see that A^{-1} is symmetric. Now considering a fixed value x such that $0 < x < 1$, we observe that for fixed $\in > 0$ and $0+\varepsilon < i/n, j/n < 1-\varepsilon$, the elements a_{ij} of A^{-1} may be approximated asymptotically by $r^{|j-i|}/(3p-4h+r)$.

Further, it may be seen that (see[11])

$$\sum \frac{r^{|j-i|}}{(3p-4h+r)} = \frac{(1+r)}{(1-r)(3p-4h+r)}$$

Where $r = 2\{2(2p)^{1/2}(p-2h)^{1/2} - (3p-4h)\}/(p-4h)^2$

We thus prove the following.

THEOREM 3.1. For a fixed $\in > 0$ and $0+\in < i/n, j/n < 1-\in$, the coefficient matrix A of (3.3) is invertible and the elements a_{ij} of A^{-1} can be approximated asymptotically by $r^{|j-i|}/(3p-4h+r$ and row max norm of its inverse, that is

$$\left\|A^{-1}\right\| \le \frac{(1+r)}{(1-r)(3p-4h+r} = K_1 \text{ (say)} \qquad (3.7)$$

where $r = 2\left[2(2p)^{1/2}(p-2h)^{1/2} - (3p-4h\right]/(p-4h)^2$.

REMARK 3.1. In studies concerning discrete splines smaller value of h have special significance for the simple reason that discrete splines reduce to continuous splines as $h \to 0$.

ERROR BOUND

For a given h>0, we introduce the set

$$R_{h0} = \{x_0 + jh : j \text{ is an integer}\}$$

and define a discrete interval as follows.

$$[0,1]_h = [0,1] \cap R_{h0}$$

In this section, we shall estimate the error bounds $e(x,h) = s(x,h) - f(x)$ over the discrete interval $[0,1]_h$. As usual the advantage in the following convergence theorem is that we do not require of its proof any smoothing condition for the function. We shall need the following Lemma [8]

LEMMA 4.1. Let $\{a_i\}$ and $\{b_j\}$ be given sequence of non-negative real numbers such that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Then for any real valued function f defined on a discrete interval $[0,1]_h$, we have

$$\left|\sum_{i=1}^{m} a_i[x_{i0}, x_{i1}, ..., x_{ik}]f - \sum_{j=1}^{n} b_j[y_{j0}, y_{j1}, ...y_{jk}]f\right| \le w\left(D_h^{\{k\}} f, |1-kh|\right)\sum a_i / k!$$

where $x_{ik}, y_{jk} \in [0,1]_h$ for relevant values of i, j, k.

In order to find the bound of $e(x)$, we substitute the value of c_i from equation (2.3) in equation (2.2) to get

$$8(p-2h)x(x,h)=$$

$$4(x-x_{i-1}-h)^2 M_i - 4(x_i - x - h)^2 M_{i-1} + (p-2h)(8f(t_i) - (p-2h)(M_i - M_{i-1})) \quad (4.1)$$

Now replacing M_i by $D_h^{\{1\}}e(x_i - h)$ and $s(x,h)$ by $e(s,h)$ in equation (4.1) we wee that it can be written in the form

$$8(p-2h)e(x,h)=$$

$$\left[4(x-x_{i-1}-h)^2 - (p-2h)^2\right]D_h^{\{1\}}e(x_i - h) - \left[4(x_i - r - h)^2 - (p-2h)^2\right]D_h^{\{1\}}e(x_{i-1} - h) + R_i(f) \quad (4.2)$$

where $R_i(f) = 8(p-2h)f(t_i) + \left[4(x-x_{i-1}-h)^2 - (p-2h)^2\right]D_h^{\{1\}} f(x_i - h) -$

$\left[4(x_i - x - h)^2 - (p-2h)^2\right]D_h^{\{1\}} f(x_{i-1} - h) - 8(p-2h)f(x).$

It may be seen easily that $R_i(f)$ can be written in the following form of divided difference.

$R_i(f) = (4(x-x_{i-1}-h)^2 - (p-2h)^2)[x_i - 2h, x_i]f - (4(x_i - x - h)^2 -$

$(p-2h)^2)[x_{i-1} - 2h, x_{i-1}]f + 4(x_i - x - h)^2[x, t_i]f - 4(x - x_{i-1} - h)^2[x, t_i]f.$

Thus

$$|R_i(f)| = \left|\sum_{i=1}^{3} a_i[x_{i0}, x_{i1}]f - \sum_{j=1}^{3} b_j[y_{j0}, y_{j1}]f\right|,$$

where $a_1 = b_1 = 4(x - x_{i-1} - h)^2, a_2 = b_2 = 4(x_i - x - h)^2,$

$a_3 = b_3 = (p-2h)^2, x_{10} = y_{30} = x_i - 2h, x_{11} = y_{31} = x_i, x_{20} = y_{10} = x,$

$x_{21} = y_{11} = t_i, x_{30} = y_{20} = x_{i-1} - 2h, x_{31} = y_{21} = x_{i-1}.$

Clearly $\sum a_i = \sum b_i$ and threfore applying Lemma 4.1 for $m = n = 3$ and $k = 1$, we have

$$|R_i(f)| \le (5p^2 + 12h^2 - 12hp)w(D_h^{\{1\}} f, p) \quad (4.3)$$

We now proceed to obtain an upper bound of $D_h^{\{1\}}e(x_i-h)$. For this we replace M_i by $D_h^{\{1\}}e(x_i-h)$ in equation (3.3) to get

$$A\left(D_h^{\{1\}}e(x_i-h)\right)=(F_i)-A\left(D_h^{\{1\}}f(x_i-h)\right)=(T_i(f)) \text{ (say), i- 1,2,...., n-1.} \quad (4.4)$$

Now, it may be seen that $(T_i(f))$ is written in the form

$$|(T_i(f))|=\left|\sum_{I=1}^{1}a_i[x_{i0},x_{i1}]f-\sum_{j=1}^{3}b_j[y_{j0},y_{i1}]f\right|$$

where $a_1=4(p-2h), b_1=b_3=(p-4h)/2, b_2=(3p-4h), x_{10}=t_i,\ x_{11}=t_{i+1}$,

$y_{10}=x_{i-1}-2h, y_{11}=x_{i-1}, y_{20}=x_i-2h, y_{21}=x_i, y_{30}=x_{i+1}-2h$, and $y_{31}=x_{i+1}$.

Clearly it is verified that $\sum a_i=\sum b_j$. Therefore, applying Lemma 4.1 again for m=1, n=3 and k=1, we get

$$|(T,(f))|\le 4(p-2h)w\left(D_h^{\{1\}}f,p\right) \quad (4.5)$$

Thus, using (3.7) and (4.5) in (4.4), we have

$$\left\|D_h^{\{1\}}e(x_i-h)\right\|\le K_2 w\left(D_h^{\{1\}}f,p\right) \quad (4.6)$$

Where $K_2=4K_1(p-2h)$.

Thus, using (4.3) and (4.6) in (4.2) we have,

$$\|e(x,h)\|\le K(p,h)w\left(D_h^{\{1\}}f,p\right) \quad (4.7)$$

Where $K(p,h)=\left(\left(6p^2-8ph\right)K_2+\left(5p^2+12h^2=12hp\right)\right)/8(p-2h)$.

We thus prove the following.

THEOREM 4.1. Suppose $s(x,h)\in D(2,P,h)$ is a discrete periodic quadratic spline interpolant of a 1-periodic function f satisfying the interpolatory condition (2.1). Then over the discrete interval $[0,1]_h$,

$$\|e(x,h)\|\le K(p,h)w\left(D_h^{\{1\}}f,p\right) \quad (4.8)$$

Where $K(p,\ h)$ is that function of p and h defined earlier. $w(f,p)$ is modulus of continuity and ||.|| is the discrete norm.

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