Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

On Two Characterizations of COTS With Endpoints

Devender Kumar Kamboj

Govt. College Gharaunda (Karnal), Haryana, 132114, India
E-mail: kamboj.dev81@rediffmail.com

ABSTRACT

In this paper, we find two characterizations of COTS with endpoints. Using these results, two characterizations of the closed unit interval are also obtained.

AMS SUBJECT CLASSIFICATION. Primary 54F05, 54F15

KEY WORDS AND PHRASES. Connected space, non-cut point, regular closed subset, cut point convex set, R(i) subset, COTS with endpoints

***Corresponding author**

Devender Kumar Kamboj

Department of Mathematics

Govt. College Gharaunda (Karnal)

Haryana, 132114, India

E-mail: kamboj.dev81@rediffmail.com

1. INTRODUCTION

Topological spaces are assumed to be connected for any consideration of cut points. The concept of COTS (= connected ordered topological space), defined by Khalimsky, Kopperman and Meyer[1], does not require any separation axiom. It is shown in[2] that an H(i) connected topological space with exactly two non-cut points is a COTS with endpoints. Papers[3-7] introduce several classes of topological spaces, whose members are COTS with endpoints. By space we mean topological space.

In this paper, we find two other characterizations of COTS with endpoints. Notation and definitions are given in Section 2. The main results of the paper appear in Section 3. In Section 3, we prove that if a connected space X has an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H\cup(X - ctX)$, then there is no proper cut point convex set of X containing all non-cut points. Thus this result generalizes Theorem 4.2 of [7]. Using this result, we prove that if a T_1 separable connected and locally connected space X has at most two non-cut points and an R(i) subset $H\cup(X - ctX)$ such that there is no proper regular closed, cut point convex set of X containing H, then X is homeomorphic to the closed unit interval. Some other characterizations of COTS with endpoints and the closed unit interval are also obtained.

2. NOTATION AND DEFINITIONS

For notation and definitions, we shall mainly follow [7]. For completeness, we have included some of the standard notation and definitions.

Let X be a space. A filter base γ on X is said to be **fixed** if $\cap\{A : A\in \gamma\} \neq \phi$. An **o-filter base** on X is a filter base whose members are open subsets of X. An o-filter base γ is a **regular o-filter base** if each member of γ contains the closure of some member of γ. X is called **R(i)**[8] if every regular o-filter base on X is fixed. Let $Y \subset X$. Y is **R(i)** if Y, as a subspace of X, is R(i). X is called $T_{1/2}$ [1] if every singleton set of X is either open in X or closed in X.

Let X be a space. Let $x \in X$, x is called a **cut point** if there exists a separation of $X - \{x\}$, if x is not a cut point of X, x is called a non-cut point of X. **ctX** is used to denote the set of all cut points of X. A separation A|B of $X - \{x\}$ is denoted by $A_x|B_x$ if the dependence of the separation on x is to be specified. A_x mentioned is a separating set out of $A_x|B_x$. $(A_x)^{+x}$ is used for the set $A_x \cup \{x\}$. For a subset Y of $X - \{x\}$, $x \in ctX$, if $Y \subset A_x$, then $A_x(Y)$ is used to denote the separating subset A_x, $(A_x(Y))^{+x}$ is used for the set $A_x(Y) \cup \{x\}$. If $Y = \{a\}$, we write $A_x(a)$ for $A_x(Y)$, $(A_x(a))^{+x}$ for $(A_x(Y))^{+x}$. Let $a, b \in X$. For $a \neq b$, $x \in ctX - \{a, b\}$, is said to be a **separating point** between a and b, or x **separates** a and b if there exists a separation $A_x|B_x$ of $X - \{x\}$, with $a \in A_x$ and $b \in B_x$. **S(a, b)** is used to denote the set of all separating points between a and b. For $x \in S(a, b)$, we write $X - \{x\} = A_x(a) \cup B_x(b)$ for a separation $A_x|B_x$ of $X - \{x\}$. If we adjoin the points a and b to S(a, b), then the new set is denoted by **S[a, b]**. If a = b, then by notation **S[a, b] = {a}.** X is called a **space with endpoints** if there exist a and b in X such that X = S[a, b]. A subset Y of X is called **cut point convex**[6] if, whenever $a, b \in Y$, $S[a, b] \subset Y$. X is called a **connected ordered topological space (COTS)**[1] if it is connected and has the property: if Y is a three-point subset of X, then there is a point x in Y such that Y meets two connected components of $X - \{x\}$. A topological property P is said to be **cut point hereditary**[4] if whenever a space X has the property P, then, for every $x \in ctX$ there exists some separation $A_x|B_x$ of $X - \{x\}$ such that each of $(A_x)^{+x}$ and $(B_x)^{+x}$ has the property P.

3. COTS WITH ENDPOINTS

The following result generalizes Theorem 4.2 of [7].

Theorem. 3.1. If a connected space X has an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$, then there is no proper cut point convex set of X containing all non-cut points.

Proof. If $X - ctX$ is empty, then the result follows by Theorem 4.2 of [7]. Now suppose that $X - ctX \neq \phi$. Assume the contrary, and let Y be a proper cut point convex set of X containing all the non-cut points of X. As $X - Y \subset ctX$, by Theorem 3.7 of [7], there exists an infinite chain α of proper connected regular closed sets of the form $(A_x(Y))^{+x}$ where $x \in X - Y$, x a closed point of X, covering X. Since H is R(i), $H \cap B_x = \phi$ for some $(A_x)^{+x} \in \alpha$ using Lemma 4.1 of [7]. Therefore $H \cup (X - ctX) \subset A_x(Y)^{+x}$. This gives a contradiction to the given condition. The proof is complete.

Theorem. 3.2. If X is a connected space having an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$, then X has at least two non-cut points.

Proof. Use Theorem 3.1.

Theorem. 3.3. A connected space X has at most two non-cut points and an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$ iff X is a COTS with endpoints.

Proof. If X has an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$, then, by Theorem 3.2, X has at least two non-cut points. Therefore, by the given condition, X has exactly two non-cut points, say, a and b. Let $x \in X - \{a,b\}$. Then $x \in ctX$. By Lemma 2.1 of [7], each of $(A_x)^{+x}$ and $(B_x)^{+x}$ is connected. By Theorem 3.1, there is no proper cut point convex set of X containing $X - ctX$. But $X - ctX = \{a,b\}$. So $a \in A_x$ and $b \in B_x$ or conversely. This implies that $x \in S(a,b)$. Hence $X = S[a,b]$. Now by Theorem 3.2 of [3], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore by Lemma 3.8(i) of [5], there is no proper connected subset of X containing {a,b}. This implies that there is no proper regular closed, connected subset of X containing {a,b} = H. Also {a,b} = X – ctX. The proof is complete.

The following remark follows using Proposition 2.9 of [1] and Theorem 3.3.

Remark. 3.4. If X is a nontrivial connected space having at most two non-cut points and an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$, then X is $T_{1/2}$.

Theorem. 3.5. If a connected space X having at most two non-cut points and an R(i) subset H such that there is no proper regular closed, connected subset of X containing $H \cup (X - ctX)$ is locally connected, then X is a compact COTS with endpoints.

Proof. By Theorem 3.3, X is a COTS with endpoints. Now the theorem follows by Theorem 4.4 of [3].

Theorem. 3.6. If a T_1 separable connected and locally connected space X has at most two non-cut points and an R(i) subset $H \cup (X - ctX)$ such that there is no proper regular closed, connected subset of X containing H, then X is homeomorphic to the closed unit interval.

Proof. By Theorem 3.5, X is a compact COTS with endpoints. X being a T_1 COTS is Hausdorff, using Proposition 2.9 of [1]. Since a separable compact connected Hausdorff space with exactly two non-cut points is homeomorphic to the closed unit interval (see Theorem 122 of [9]), X is homeomorphic to the closed unit interval.

The following result generalizes Theorem 4.11 of [7].

Theorem. 3.7. Let X be a connected space having a cut point hereditary property P. If X has an R(i) subset H such that there is no proper regular closed, connected subset of X, having property P and containing $H \cup (X - ctX)$, then there is no proper cut point convex set of X containing all non-cut points.

Proof. If X – ctX is empty, then the result follows by Theorem 4.11 of [7]. Now suppose that $X - ctX \neq \phi$. Let Y be a proper cut point convex set of X containing all the non-cut points of X. As $X - Y \subset ctX$, by Lemma 3.11 of [7], there exists an infinite upwards chain α of proper connected regular closed subsets of the form $(A_x(Y))^{+x}$, $x \in X - Y$, x a closed point of X and $(A_x(Y))^{+x}$ has property P, covering X. Since H is R(i), $H \cap B_x = \phi$ for some $(A_x)^{+x} \in \alpha$ using Lemma 4.1 of [7]. Therefore $H \cup (X - ctX) \subset A_x(Y)^{+x}$. This gives a contradiction to the given condition. The proof is complete.

The following result is the non-cut point existence theorem for another subclass of connected spaces having a cut point hereditary property.

Theorem. 3.8. Let X be a connected space having a cut point hereditary property P. If X has an R(i) subset H such that there is no proper regular closed, connected subset of X, having property P and containing $H\cup(X - ctX)$, then X has at least two non-cut points of X.

Proof. If the result is not true, then exists some $x \in X$ such that $X - ctX \subset \{x\}$ which is contrary to Theorem 3.7. Then there are at least two non-cut points of X.

We note that the following result follows on the lines of the proof of Theorem 3.3 because, for a cut point x of a connected space X having a cut point hereditary property P, there exists a separation $A_x|B_x$ of $X - \{x\}$ such that each of $(A_x)^{+x}$ and $(B_x)^{+x}$ has the property P, and Theorems 3.8 and 3.7 are applicable, in place of Theorems 3.2 and 3.1 respectively.

Theorem. 3.9. Let X be a connected space having a cut point hereditary property P. Then X is a COTS with endpoints if and only if X has at most two non-cut points and an R(i) subset H such that there is no proper regular closed, connected subset of X, having property P and containing $H\cup(X - ctX)$.

Theorem. 3.10. Let X be a connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has an R(i) subset H such that there is no proper regular closed, connected subset of X, having property P and containing $H\cup(X - ctX)$, then X is a compact COTS with endpoints.

Proof. By Theorem 3.9, X is a COTS with endpoints. Now the theorem follows by Theorem 6.1(ii) of [4].

Theorem 3.10 coupled with Corollary 6.2(i) of [4] gives the following corollary.

Corollary. 3.11. Let X be a T_1 separable, connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has an R(i) subset H such that there is no proper regular closed, connected subset of X, having property P, containing H, then X is homeomorphic to the closed unit interval.

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