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Finite Double Integral Representation For The Polynomial Set $S_n(X, Y)$

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ABSTRACT

In the present paper, an attempt has been made to express a Finite Double Integral representation for the polynomial set $S_n(x, y)$. Many interesting new results may be obtained as particular cases on separating the parameter.

KEYWORD: Appell Function, Generalized Hyper geometric Polynomial, Integral Representation, Orthogonal Polynomial.

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1. INTRODUCTION

We define the generalized hypergeometric polynomial set $S_n(x, y)$ by means of the generating functions,

$$e^{\lambda yt}\ F\left[\begin{matrix}(G_r); \\ \lambda_1 y^{e_1} t^{e_1} \\ (H_s);\end{matrix}\right] \times F\left[\begin{matrix}(a_p);(A_h);(C_u) \\ \lambda_3 x^{e_3} t, \lambda_2 x^{e_2} y^{-e_2} t^{e_2} \\ (b_q);(B_k);(D_v)\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} S_{n_1;e_1;e_2;e_3;(H_s);(b_q);(B_k);(D_v)}^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}(x, y)t^n \qquad \ldots (1.1)$$

Where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are real and e_1, e_2, e_3 are positive integers.

The left hand side of (1.1) contains Appell function of two variables in the notation of Burchnall and Chaundy[1]

The polynomial set contains a number of parameters, for simplicity, we shall denote

$S_{n;e_1;e_2;e_3;(H_s)(b_q);(B_k);(D_v)}^{\lambda;\lambda_1;\lambda_2;\lambda_3;(G_r);(a_p);(A_h);(C_u)}(x, y)$ by $S_n(x, y)$.

Where n denote the order of the polynomial set.

After little simplification (1.1) gives

$$S_n(x, y) = \sum_{\substack{m,m_1,m_2>0 \\ m+e_1m_1+e_2m_2\leq 0}} \frac{\Delta(m_1, m_2)}{(n-m-e_1m_1-e_2m_2)!} \qquad \ldots (1.2)$$

Where $\Delta(m_1, m_2) = \dfrac{[(a_p)]_{n-m-e_1m_1-(e_2-1)m_2}}{[(b_q)]_{n-m-e_1m_1-(e_2-1)m_2}}$

$$\times \frac{[(A_h)]_{n-m-e_1m_1-e_2m_2}[(G_r)]_{m_1}[(C_u)]_{m_2}}{[(B_k)]_{n-m-e_1m_1-e_2m_2}[(H_s)]_{m_1}[(D_v)]_{m_2}}$$

$$\times \frac{x^{m_2e_2}\lambda^m\lambda_1^{m_1}\lambda_2^{m_2}(\lambda_3 x^{e_3})^{n-m-e_1m_1-e_2m_2}\ y^{m+e_1m_1+e_2m_2}}{m!\quad m_1!\quad m_2!\ (n-m-e_1m_1-e_2m_2)!}$$

The polynomial set $S_n(x, y)$ happens to the generalization of as many as forty-one orthogonal and non-orthogonal polynomials.

2. NOTATIONS

(i) $(m) = 1, 2, 3, \ldots\ldots, m$.

(ii) $(A_p) = A_1, A_2, A_3, \ldots\ldots A_p$.

(iii) $[(A_p)] = A_1 A_2 A_3 \ldots\ldots A_p$.

(iv) $[(A_p)]_n = (A_1)_n (A_2)_n (A_3)_n \ldots\ldots (A_p)_n$.

(v) $\Delta(a,b) = \frac{b}{a}, \frac{b+1}{a}, \ldots\ldots \frac{b+a-1}{a}$.

(vi) $\Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b)$.

(vii) $\Gamma_* \Gamma_* (a+b) = \Gamma(a+b)\Gamma(a+b)$.

(viii) $K = \frac{[(a_p)]_n [(A_h)]_n (\lambda_3 x^{e_3})^n}{[(b_q)]_n [(B_k)]_n \quad n!}$

3. THEOREM

For $e_2 > 1$, we achieve

$$S_n(x,y) = \frac{\Gamma(\alpha+\beta+1)}{(1-x)^{\alpha+\beta}\ \Gamma(\alpha)\Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1}$$

$$\times \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \times \frac{[(a_p)]_{n-m-e_1m_1} [(A_h)]_{n-m-e_1m_1} [(G_r)]_{m_1}}{[(b_q)]_{n-m-e_1m_1} [(B_k)]_{n-m-e_1m_1} [(H_s)]_{m_1}}$$

$$\times \frac{\lambda^m \lambda_1^{m_1} (\lambda_2 x^{e_2})^{n-m-e_1m_1} \quad y^{m+e_1m_1}}{m! \quad m_1! \quad (n-m-e_1m_1)}$$

$$\times F \left[\begin{matrix} \Delta(e_2; -n+m+e_1m_1), \Delta(e_2-1; 1-(b_q)-n+m+e_1m_1) \\ \Delta(e_2; 1-(B_k)-n+m+e_1m_1), (C_u), \Delta(2e_2; \alpha+\beta+1); \\ \dfrac{\lambda_2 x^{e_2} \{-e_2\}^{e_2(q-p+h-k+1)} \{-(e_2-1)\}^{(e_2-1)(q-p)}}{(\lambda_3 x^{e_3} y)^{e_2} (1-x)^{2e_2}} \\ \Delta(e_2-1; 1-(a_p)-n+m+e_1m_1), \\ \Delta(e_2; 1-(A_h)-n+m+e_1m_1), (D_u); \alpha, \beta; \end{matrix} \right] dydz \qquad \ldots$$

(3.1)

Where $x + y < 1$.

$$\text{Proof: I} = \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1} \sum_{m=0}^{n} \sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]} \times \frac{[(a_p)]_{n-m-e_1m_1}}{[(b_q)]_{n-m-e_1m_1}}$$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1}\left[(G_r)\right]_{m_1}}{\left[(B_k)\right]_{n-m-e_1m_1}\left[(H_s)\right]_{m_1}}\frac{\lambda^m}{m!}\frac{\lambda_1^{m_1}}{m_1!}\frac{\left(\lambda_2 x^{e_2}\right)^{n-m-e_1m_1}}{(n-m-e_1m_1)}y^{m+e_1m_1}$$

$$\times F\left[\begin{array}{c}\Delta(e_2;-n+m+e_1m_1),\Delta\left(e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right),\\ \Delta\left(e_2;1-(B_k)-n+m+e_1m_1\right),(C_u),\Delta(2e_2;\alpha+\beta+1);\\ \dfrac{\lambda_2 x^{e_2}\{-e_2\}^{e_2(q-p+h-k+1)}\{-(e_2-1)\}^{(e_2-1)(q-p)}}{\left(\lambda_3 x^{e_3}y\right)^{e_2}(1-x)^{2e_2}}\\ \Delta\left(e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right),\\ \Delta\left(e_2;1-(A_h)-n+m+e_1m_1\right),-(D_u),\ \alpha,\ \beta;\end{array}\right]dydz$$

$$=\int_0^{1-x}\int_0^{1-x-y}y^{\alpha+m_2-1}z^{\beta+m_2-1}\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\times\frac{\left[\left(a_p\right)\right]_{n-m-e_1m_1}}{\left[\left(b_q\right)\right]_{n-m-e_1m_1}}$$

$$\times\frac{\left[(A_h)\right]_{n-m-e_1m_1}\left[(G_r)\right]_{m_1}}{\left[(B_k)\right]_{n-m-e_1m_1}\left[(H_s)\right]_{m_1}}\frac{\lambda^m}{m!}\frac{\lambda_1^{m_1}}{m_1!}\frac{\left(\lambda_2 x^{e_2}\right)^{n-m-e_1m_1}}{(n-m-e_1m_1)}y^{m+e_1m_1}$$

$$\times\frac{\Delta_{m_2}\left[e_2;-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2;1-(A_h)-n+m+e_1m_1\right]}$$

$$\times\frac{\Delta_{m_2}\left[e_2;1-(B_k)-n+m+e_1m_1\right]\left[(C_u)\right]_{m_2}}{\left[(D_u)\right]_{m_2}}\frac{\Delta_{m_2}(2e_2;\alpha+\beta+1)}{m_2!\ (\alpha)_{m_2}\ (\beta)_{m_2}}$$

$$\times\frac{\left(\lambda_2 x^{e_2}\right)^{m_2}(-e_2)^{e_2(q-p+h-k+1)m_2}\{-(e_2-1)\}^{(e_2-1)(q-p)}dydz}{(1-x)^2\left(\lambda_3 x^{e_3}y\right)^{e_2m_2}}$$

$$=\sum_{m=0}^{n}\sum_{m_1=0}^{\left[\frac{n-m}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-m-e_1m_1}{e_2}\right]}\times\frac{\left[\left(a_p\right)\right]_{n-m-e_1m_1}\left[(A_h)\right]_{n-m-e_1m_1}}{\left[\left(b_q\right)\right]_{n-m-e_1m_1}\left[(B_k)\right]_{n-m-e_1m_1}}$$

$$\times\frac{\left[(G_r)\right]_{m_1}}{\left[(H_s)\right]_{m_1}}\frac{\lambda^m}{m!}\frac{\lambda_1^{m_1}}{m_1!}\frac{\left(\lambda_2 x^{e_2}\right)^{n-m-e_1m_1}}{(n-m-e_1m_1)}y^{m+e_1m_1}$$

$$\times\frac{\Delta_{m_2}\left[e_2;-n+m+e_1m_1\right]\Delta_{m_2}\left[e_2-1;1-\left(b_q\right)-n+m+e_1m_1\right]}{\Delta_{m_2}\left[e_2-1;1-\left(a_p\right)-n+m+e_1m_1\right]}$$

$$\times \frac{\Delta_{m_2}\left[e_2;1-(B_k)-n+m+e_1m_1\right]\left[(C_u)\right]_{m_2}}{\Delta_{m_2}\left[e_2;1-(A_h)-n+m+e_1m_1\right]\left[(D_u)\right]_{m_2}}$$

$$\times \frac{\Delta_{m_2}\left[2e_2;\alpha+\beta+1\right]\left(\lambda_2 x^{e_2}\right)^{m_2}(-e_2)^{e_2(q-p+h-k+1)m_2}}{(\alpha)_{m_2}(\beta)_{m_2}(1-x)^{2m_2}\left(\lambda_3 x^{e_3} y\right)^{e_2 m_2}}$$

$$\times \frac{(1-x)^{\alpha+\beta+2m_2}\Gamma(\alpha+m_2)\Gamma(\beta+m_2)\left\{(e_2-1)\right\}^{(e_2-1)(q-p)m_2}}{\Gamma(\alpha+\beta+1+2m_2)}$$

$$= \frac{(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} S_n(x,y)$$

On using Orthogonal Polynomials by Gabor Szego[2]

$$\int_0^{1-x}\int_0^{1-x-y} y^{m-1} z^{n-1}\,dydz = \frac{(1-x)^{m+n}\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}$$

Where $x+y<1$

Particular Cases of (3.1)

Separating the term corresponding to $\lambda = 0$ and putting $r = 0 = s = \lambda_1$ in (3.1), we obtain a number of results on specializing the remaining parameters :

(i) Hermit polynomials:

On Putting $p = 0 = q = h = k = u = v;\ m = 1 = m_1 = e_1 = e_3 = \lambda;\ \lambda_3 = 1 = e_2;\ \lambda_2 = -1,\ y = x,$ We get

$$H_n(x) = \frac{(2x)^n\Gamma(\alpha+\beta+1)}{n!(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^{1-x}\int_0^{1-x-y} y^{\alpha-1}z^{\beta-1}$$

$$\times F\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \Delta(2;\alpha+\beta+1); \\ \qquad\qquad\qquad \frac{-1}{(1-x)^2} \\ \alpha,\ \beta; \end{matrix}\right] dydz$$

(ii) Legendre polynomials:

If we take $p = 0 = q = h = k = u;\ v = 1 = m = m_1 = e_1 = \lambda_2 = e_3;\ D_1 = 1;\ \lambda_3 = 1;\backslash$

$y = 2x,\ e_2 = 2,$ and $\frac{x}{\sqrt{x^2-1}}$ for x, we get

$$P_n(x) = \frac{x^n\Gamma(\alpha+\beta+1)}{n!(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^{1-x}\int_0^{1-x-y} y^{\alpha-1}z^{\beta-1}$$

$$\times F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\Delta(2;\alpha+\beta+1); \\ \qquad\qquad \frac{x^2-1}{x^2(1-x)^2} \\ 1,\alpha,\ \beta;\end{matrix}\right]dydz$$

ii. Jackson polynomials: On taking $p=0=q=h=k=u=v;\ e_2=2,\ \lambda_3=4;\ \lambda_2=-4,\ e_2=4,\ e_3=1=\lambda=\lambda_1;\ y=x$

We achieve

$$\mathfrak{H}_n(x)=\frac{2^{2n}x^n\Gamma(\alpha+\beta+1)}{n!(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^{1-x}\int_0^{1-x-y}y^{\alpha-1}z^{\beta-1}$$

$$\times F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\Delta(2;\alpha+\beta+1); \\ \qquad\qquad \frac{-1}{x^2(1-x)^2} \\ \alpha,\ \beta;\end{matrix}\right]dydz$$

(iv) If we set $p=0=q=h=k;\ u=1=v=e_3;\ y=x;\ e_2=m;\ \lambda_2=\mu;\ \lambda_3=v;\ C_1=a_r,\ D_1=b_s,$

We have

$$A_n(x)=\frac{(vx)^n\Gamma(\alpha+\beta+1)}{n!(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^{1-x}\int_0^{1-x-y}y^{\alpha-1}z^{\beta-1}$$

$$\times F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\Delta(2;\alpha+\beta+1),(a_r); \\ \qquad\qquad \mu\left(\frac{-m}{vx}\right)^m\frac{1}{(1-x)^2} \\ \alpha,\ \beta;(b_s)\end{matrix}\right]dydz$$

Where A_n (x) are the generalized by Panda[3]

(v) On making the substitution $p=0=q=h=k=u=v;\ e_3=1=\lambda_3;\ \lambda_2=h,\ y=x,$

We have

$$g_n^m(x,h)=\frac{x^n\Gamma(\alpha+\beta+1)}{n!(1-x)^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^{1-x}\int_0^{1-x-y}y^{\alpha-1}z^{\beta-1}\times F\left[\begin{matrix}\Delta(m;-n),\Delta(2;\alpha+\beta+1), \\ \qquad\qquad \frac{h}{(1-x)^2}\left(\frac{-m}{x}\right)^m \\ \alpha,\ \beta;\end{matrix}\right]dydz$$

Where $g_n^m(x,h)$ are the generalized polynomials defined by Gould-Hopper[4]

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