Research article **Available online www.ijsrr.org** **ISSN: 2279–0543**

International Journal of Scientific Research and Review

An Insight into the Indefinite Super Hyperbolic GKM Algebra $SHGGH_2^{(3)}$

G.Uthra and M.Priyanka*

Department of Mathematics,Pachaiyappa's College, Chennai-600 030 , Tamilnadu, India
E-mail: priyankamahalingam31@gmail.com

ABSTRACT

In the paper the indefinite Super Hyperbolic Generalized Kac-Moody algebra $SHGGH_2^{(3)}$ obtained as an extension of $H_2^{(3)}$ is studied. The connectedDynkin diagrams which are non-isomorphic and associated with this family $SHGGH_2^{(3)}$ is completely classified. Some of the basic properties of real, imaginary, isotropic, strictly imaginary, special imaginary, purely imaginary roots are discussed.

KEY WORDS: Generalized Generalized Cartan Matrix, Dynkin diagrams, real root, imaginary root, isotropic, Super hyperbolic.

AMS MSC 2010 Code: 17B67

***Corresponding author**

M. Priyanka

Department of Mathematics,

Pachaiyappa's College, Chennai-600 030 ,

Tamilnadu, India

E-mail: priyankamahalingam31@gmail.com

1. INTRODUCTION:

Borcherds constructed the Generalized Kac-Moody algebra (abbreviated as GKM algebra)[1]. Bennett[2] and Caperson[3] introduced the special and strictly imaginary roots of Kac-Moody algebras. Stanumoorthy et al. compiled the existence and non existence of purely imaginary, strictly imaginary and special imaginary roots for finite, affine and hyperbolic types and also computed the root multiplicities for the same[5,6,7,8,9]. Xinfang Song and et al.[10,11]determined the root structure and root multiplicity for an infinite GKM algebra.

In this paper, we consider the Super Hyperbolic GKM algebras $SHGGH_2^{(3)}$ obtained as an extension of $H_2^{(3)}$ with one imaginary simple root. The complete classification of connected non-isomorphic Dynkin diagrams associated with $SHGGH_2^{(3)}$ is obtained. The properties of real, imaginary, isotrophic, strictly imaginary, purely imaginary and special imaginary roots for the GKM algebra $SHGGH_2^{(3)}$ are discussed.

2. PRELIMINARIES

Basic definitions and notations are defined as in[2]

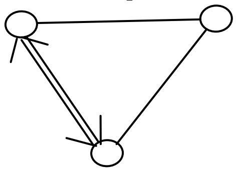
Definition 2.1[1]: In GKM algebras the Dynkin diagrams is defined as follows:To every GGCM A is associated a Dynkin diagram S(A) defined as follows : S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|,|a_{ji}|\}$ number of lines if $a_{ij}.a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}|>1$. If $a_{ij}.a_{ji}> 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|,|a_{ji}|)$ of integers. If $a_{ii} = 2$, i^{th} vertex will be denoted by a white circle and if $a_{ii} = 0$, i^{th} vertex will be denoted by a crossed circle. If $a_{ii} = -k$, $k > 0$, i^{th} vertex will be denoted by a white circle with $-k$ written above the circle within the parenthesis.

Definition 2.2[4]:A GGCM $A = (a_{ij})_{i,j=1}^{n}$ is of SuperHyperbolic type (abbreviated as SH type) if A is not of hyperbolic type and any indecomposable proper principal submatrix of A is of finite, affine or hyperbolic type.

3. COMPLETE CLASSIFICATION OF DYNKIN DIAGRAMS OF GKM ALGEBRAS $SHGGH_2^{(3)}$ WITH ONE IMAGINARY SIMPLE ROOT

In this section, we consider the GGCM $\begin{pmatrix} -k & -p_1 & -p_2 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$, where $k, p_i, q_i \in \Re_+ \cup \{-2\} \forall i$

Proposition 3.1 : There are 1098 connected non-isomorphic Dynkin diagrams associated with the GGCM of indefinite Super hyperbolic type $SHGGH_2^{(3)}$.



Proof : The Dynkin diagramassociated with the hyperbolic family $H_2^{(3)}$ is

We extend the 4th vertex with $H_2^{(3)}$ and get all possible combinations of connected non-isomorphic Dynkin diagrams for the associated GGCM of Super hyperbolic type $SHGGH_2^{(3)}$. Here ⁓ can be represented by one of the possible 9 edges:

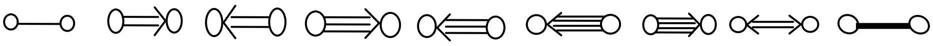
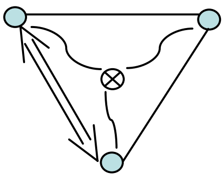
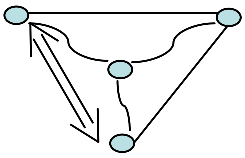
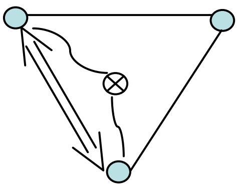
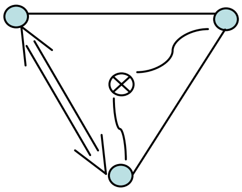
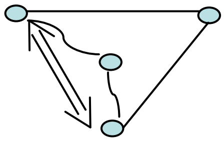
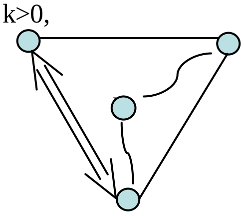
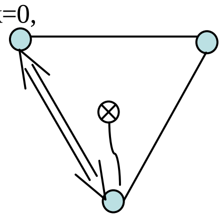
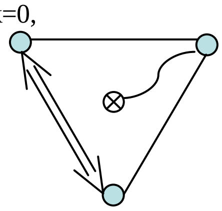
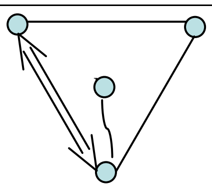
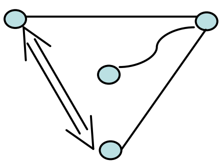


Table No. 1: Complete Classification of Dynkin Diagram of $SHGGH_2^{(3)}$

Extended Dynkin diagram of Superhyperbolic type $SHGGH_2^{(3)}$	Correspondin GGCM	Number of possible Dynkin diagrams
When k=0, 	$\begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$	In this case, we connect the fourth vertex to all the 3 vertices and there exists are 9^3 connected Dynkin diagrams in which 344 are isomorphic Dynkin diagrams. Excluding these, we get 405 non-isomorphic connected Dynkin diagrams, for both the case, when k = 0 and k > 0
When k>0, 	$\begin{pmatrix} -k & -p_1 & -p_2 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$	
When k=0, 	$\begin{pmatrix} 0 & -p_1 & 0 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$	In this case, among the 3 vertices, two of the vertices are connected with different combinations to the fourth vertex by the 9 possible edges. Therefore, in this case, the associated connected Dynkin

When k=0, 	$\begin{pmatrix} 0 & -p_1 & -p_2 & 0 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	diagrams are 2x 9^2 = 243 and excluding the 117 isomorphic Dynkin diagrams, we get 126 non-isomorphic connected Dynkin diagrams, for both the case, when k = 0 and k > 0.
When k>0, 	$\begin{pmatrix} -k & -p_1 & 0 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$	
When k>0, 	$\begin{pmatrix} -k & -p_1 & -p_2 & 0 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	
When k=0, 	$\begin{pmatrix} 0 & -p_1 & 0 & 0 \\ -q_1 & 2 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	Let us connect the fourth vertex independently to the other three vertices by the 9 possible edges. Thus, the possible number of connected Dynkin diagrams associated with $SHGGH_2^{(3)}$ is 3 x 9 = 27. But by joining the vertices 2 and 4, we get 9 isomorphic Dynkin diagrams. Thus, by deleting these isomorphic diagrams we get, 18 diagrams when k = 0 and 18 diagrams when k > 0.
When k=0, 	$\begin{pmatrix} 0 & 0 & -p_2 & 0 \\ 0 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	
When k>0, 	$\begin{pmatrix} -k & -p_1 & 0 & 0 \\ -q_1 & 2 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	
When k>0, 	$\begin{pmatrix} -k & 0 & -p_2 & 0 \\ 0 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}$	

Thus there exists 1098 types of connected, non isomorphic Dynkin diagrams associated with the GGCM of $SHGGH_2^{(3)}$.

4. PROPERTIES OF ROOTS

Consider the symmetrizable GGCM of $QHGGH_2^{(3)} = \begin{pmatrix} -k & -p_1 & -p_2 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$ with the conditions $p_1 q_2 = p_2 q_1 \ and \ p_3 q_1 = p_1 q_3$ where $k, p_i, q_i \in \Re_+ \cup \{-2\} \forall i$.

We have $\prod = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\prod^{re} = \{\alpha_2, \alpha_3, \alpha_4\} \ and \ \prod^{im} = \{\alpha_1\}$.

The non-degenerate symmetric bilinear form are given as,

$(\alpha_1, \alpha_1) = -kq_1, (\alpha_1, \alpha_2) = -q_1 p_1, (\alpha_1, \alpha_3) = -p_1 q_2, (\alpha_1, \alpha_4) = -p_1 q_3, (\alpha_2, \alpha_2) = 2p_1,$

$(\alpha_2, \alpha_3) = -p_1, (\alpha_2, \alpha_4) = -2p_1, (\alpha_3, \alpha_3) = 2p_1, (\alpha_3, \alpha_4) = -p_1, (\alpha_4, \alpha_4) = 2p_1$

The fundamental reflections are computed as follows:

$r_2(\alpha_1) = \alpha_1 + q_1 p_1 \alpha_2, r_2(\alpha_2) = -\alpha_2, r_2(\alpha_3) = \alpha_3 + \alpha_2, r_2(\alpha_4) = \alpha_4 + 2\alpha_2,$

$r_3(\alpha_1) = \alpha_1 + q_2 p_1 \alpha_3, r_3(\alpha_2) = \alpha_2 + \alpha_3, r_3(\alpha_3) = -\alpha_3, r_3(\alpha_4) = \alpha_4 + \alpha_3,$

$r_4(\alpha_1) = \alpha_1 + q_3 p_1 \alpha_4, r_4(\alpha_2) = \alpha_2 + 2\alpha_4, r_4(\alpha_3) = \alpha_3 + \alpha_4, r_4(\alpha_4) = -\alpha_4$

Here, $\Delta_+^{im} = \bigcup_{w \in W} w(K)$ where K is given by

$K = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 / k_1 \in \mathrm{N}, k_2, k_3, k_4 \in \mathrm{Z}_+,$
$2k_2 \le q_1 k_1 + k_3 + 2k_4, 2k_3 \le q_2 k_1 + k_2 + k_4 \ and \ 2k_4 \le q_3 k_1 + 2k_2 + k_3 \ with$
$k_2 = 0 \Rightarrow 2k_3 \le q_2 k_1 + k_4 \ and \ 2k_4 \le q_3 k_1 + k_3, k_3 = 0 \Rightarrow 2k_2 \le q_1 k_1 + 2k_4 \ and \ 2k_4 \le q_3 k_1 + 2k_2$
$k_4 = 0 \Rightarrow 2k_2 \le q_1 k_1 + k_3 \ and \ 2k_3 \le q_2 k_1 + k_2, k_2 = k_3 = 0 \Rightarrow 2k_4 \le q_3 k_1; \ k_2 = k_4 = 0 \Rightarrow 2k_3 \le q_2 k_1$
$k_3 = k_4 = 0 \Rightarrow 2k_2 \le q_1 k_1; \ k_2 = k_3 = k_4 = 0 \Rightarrow k_1 = 1\}$

Note that, the Weyl group is infinite.

Root Properties of $SHGGH_2^{(3)}$ **:** In this section, we discuss the real, imaginary, purely imaginary, strictly imaginary and special imaginary roots of $SHGGH_2^{(3)}$.

Real Roots: All simple real roots have same length $2p_1$.

Roots of Height 2:

1) $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = -kq_1+2p_1-2p_1q_1$

Case (i): When $k \neq 0$

$$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = \begin{cases} -kq_1+2p_1-2p_1q_1 & is\ imaginary\quad if\ p_1 > q_1 \\ 0 & is\ isotropic\quad if\ p_1 = q_1 = 0 \end{cases}$$

Case (ii): When k=0 ; $(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2p_1-2p_1q_1$

$$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = \begin{cases} 2p_1-2p_1q_1 & is\ imaginary\quad if\ p_1 < q_1 \\ 0 & is\ isotropic\quad if\ p_1 = 0 \end{cases}$$

2) $(\alpha_1+\alpha_3, \alpha_1+\alpha_3) = -kq_1+2p_1-2p_1q_2$

Case (i): When $k \neq 0$

$$(\alpha_1+\alpha_3, \alpha_1+\alpha_3) = \begin{cases} -kq_1+2p_1-2p_2q_1 & is\ real\quad if\ -kq_1-2p_2q_1 < 2p_1 \\ -kq_1+2p_1-2p_1q_2 & is\ imaginary\quad if\ p_1 < q_2 \\ 0 & is\ isotropic\quad if\ p_i = q_i = 0 \end{cases}$$

Case (ii): When k=0 ; $(\alpha_1+\alpha_3, \alpha_1+\alpha_3) = 2p_1-2p_1q_2$

$$(\alpha_1+\alpha_3, \alpha_1+\alpha_3) = \begin{cases} 2p_2-2p_1q_{21} & is\ real\quad if\ p_2 \neq 0, p_1q_2 < p_2 \\ 2p_2-2p_1q_2 & is\ imaginary\quad if\ p_1q_2 > p_2 \\ 0 & is\ isotropic\quad if\ p_i\ \&\ q_i = 0\ or\ p_i = q_i = 1 \end{cases}$$

3) $(\alpha_2+\alpha_3, \alpha_2+\alpha_3) = 2p_1$ so that $\alpha_2+\alpha_3$ is a real root.

4) $(\alpha_2+\alpha_4, \alpha_2+\alpha_4) = 0$ so that $\alpha_2+\alpha_4$ is isotropic.

Similarly, the other cases of height 2 roots $\alpha_1+\alpha_4$, $\alpha_3+\alpha_4$ can be discussed.

Roots of Height 3:

1) $(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = -4kq_1+2p_1-4p_1q_1$

Case (i): When $k \neq 0$

$$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = \begin{cases} -4kq_1+2p_1-4p_1q_1 & is\ imaginary\quad if\ p_1 > q_1 \\ 0 & is\ isotropic\quad if\ p_1 = q_1 = 0 \end{cases}$$

Case (ii): When k=0 ; $(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = 2p_1-4p_1q_1$

$$(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2) = \begin{cases} 2p_1-4p_1q_1 & is\ imaginary\quad if\ p_1 \geq q_1 \\ 0 & is\ isotropic\quad if\ p_1 = 0 \end{cases}$$

2) $(\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4) = -2p_1$

$$(\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4) = \begin{cases} -2p_1 & is\ imaginary\quad if\ p_1 > 0 \\ 0 & is\ isotropic\quad if\ p_1 = 0 \end{cases}$$

3) $(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4) = (\alpha_3+2\alpha_4, \alpha_3+2\alpha_4) = 6p_1$

$$(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4) = (\alpha_3+2\alpha_4, \alpha_3+2\alpha_4) = \begin{cases} 0 & \text{is isotropic} \quad \text{if } p_1 = 0 \\ 6p_1 & \text{is real} \quad \text{if } p_1 > 0 \end{cases}$$

Similarly, the other cases of height 3 roots can be computed.

Roots of Height 4:

1) $(2\alpha_3+2\alpha_4, 2\alpha_3+2\alpha_4) = 8p_1$

$$(2\alpha_3+2\alpha_4, 2\alpha_3+2\alpha_4) = \begin{cases} 0 & \text{is isotropic} \quad \text{if } p_1 = 0 \\ 8p_1 & \text{is real} \quad \text{if } p_1 > 0 \end{cases}$$

2) $(\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4) = -kq_1-2p_1q_1-2p_2q_1-2p_3q_1-2p_1$

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4) = \begin{cases} -kq_1-2p_1q_1-2p_2q_1-2p_3q_1-2p_1 & \text{is imaginary} \quad \text{if } k \neq 0, p_i, q_i > 0 \\ -kq_1-2p_1q_1-2p_2q_1-2p_3q_1-2p_1 & \text{is imaginary} \quad \text{if } k \neq 0 \text{ and } p_i \text{ or } q_i = 0 \\ 0 & \text{is isotropic} \quad \text{if } k = 0, p_i = q_i = 0 \end{cases}$$

3) $(3\alpha_2+\alpha_3, 3\alpha_2+\alpha_3) = 14p_1$

$$(3\alpha_2+\alpha_3, 3\alpha_2+\alpha_3) = \begin{cases} 14p_1 & \text{is real} \quad \text{if } p_1 > 0 \\ 0 & \text{is isotropic} \quad \text{if } p_1 = 0 \end{cases}$$

The remaining roots of height 4 can be discussed in similar manner.

Proposition 4.1: Let A= $\begin{pmatrix} -k & -p_1 & -p_2 & -p_3 \\ -q_1 & 2 & -1 & -2 \\ -q_2 & -1 & 2 & -1 \\ -q_3 & -2 & -1 & 2 \end{pmatrix}$ be the symmetrizable GGCM of $SHGGH_2^{(3)}$, where $k, p_i, q_i \in \Re_+ \cup \{-2\} \forall i$. There exists no special imaginary root of g(A).

Proof: Suppose $\alpha = \sum_{i-1}^{n+1} k_i\alpha_i \in K \subseteq \Delta_+^{im}, k_i \in Z_+ \forall i$ be a special imaginary root of g(A).We have

$$(\alpha,\alpha) = 2p_1k_2^2 + 2p_1k_3^2 + 2p_1k_4^2 - q_1kk_1^2 - 2p_1q_1k_1k_2 - 2p_2q_1k_1k_3 - 2p_3q_1k_1k_4 - 2p_1k_2k_3$$
$$- 4p_1k_2k_4 - 2p_1k_3k_4 \quad < 0$$

Let *(α,α)=A*. By the reflection of imaginary root definition, we have

$$\left.\begin{aligned} r_\alpha(\alpha_1) = \alpha_1 + \frac{2q_1}{A}(kk_1 + p_1k_2 + p_2k_3 + p_3k_4)\alpha, r_\alpha(\alpha_2) = \alpha_2 - \frac{2p_1}{A}(2k_2 - q_1k_1 - k_3 - 2k_4)\alpha \\ r_\alpha(\alpha_3) = \alpha_3 - \frac{2p_1}{A}(2k_3 - q_2k_1 - k_2 - k_4)\alpha, r_\alpha(\alpha_4) = \alpha_4 - \frac{2p_1}{A}(2k_4 - q_3k_1 - 2k_2 - k_3)\alpha \end{aligned}\right\} \text{...(1)}$$

Then for a special imaginary root α, we have $r_\alpha(\alpha_2) = \alpha_2; r_\alpha(\alpha_3) = \alpha_3; r_\alpha(\alpha_4) = \alpha_4 \ldots$ (2)

From the above equations (1) and (2), we get

$$(2k_2 - q_1k_1 - k_3 - 2k_4)\alpha = (2k_3 - q_2k_1 - k_2 - k_4)\alpha = (2k_4 - q_3k_1 - 2k_2 - k_3)\alpha = 0$$

Then, $8k_4 = k_1(q_1 - 4q_2 + 3q_3)$ is absurd. Therefore, there exists no special imaginary root for g(A), where A is a symmetrizable $SHGGH_2^{(3)}$.

Proposition 4.2: The SH GKM algebra $SHGGH_2^{(3)}$ satisfies the purely imaginary property.

Proof: By[6], there are some GKM algebras possessing and not possessing the purely imaginary property. Here the, SH GKM algebra $SHGGH_2^{(3)}$ satisfies the purely imaginary property. Because, α_1 is an imaginary simple root and if we add any root with α_1 we get an imaginary root and also support of α contains n vertices and it is connected. Therefore, all imaginary roots are purely imaginary and hence, for any $\alpha \in \Delta_+^{im}$ and for any $\beta \in \Delta_+^{im}$ we get $\alpha + \beta \in \Delta_+^{im}$ is a root.

Example: $2\alpha_1+\alpha_2+\alpha_3$ satisfies the purely imaginary property.

(i.e). $\alpha = \alpha_1+\alpha_2$, $\beta = \alpha_1+\alpha_3$; we get $\alpha+\beta = -4kq_1+4p_1-4p_1q_1-4p_2q_1$ (where $p_1<q_1$), which satisfies the purely imaginary property.

Proposition 4.3: The SH GKM algebra $SHGGH_2^{(3)}$ satisfies the strictly imaginary property.

Proof: Since support of α is connected, the sum or difference of any combination of α_i (i=1,2,3&4) is a root. Hence, for any $\alpha \in \Delta^{re}$ and for any $\gamma \in \Delta_+^{im}$ we get $\alpha + \gamma$ is a root. Therefore, the Super hyperbolic generalized generalized Kac-Moody algebra $SHGGH_2^{(3)}$ has the strictly imaginary property.

Example: $\alpha_1+\alpha_2+\alpha_3$ satisfies the strictly imaginary property.

(i.e). $\alpha = \alpha_2$, $\gamma = \alpha_1+\alpha_3$; we get $\alpha+\gamma = -kq_1+2p_1-2p_1q_1-2p_2q_1$ is a root, which satisfies the strictly imaginary property.

CONCLUSION:

In this work, the Dynkin diagrams are completely classified and some properties of roots are obtained for the family $SHGGH_2^{(3)}$. Further, we can compute the root multiplicities forvarious families of GKM algebras.

REFERENCES:

1. Borcheds RE. Generalized Kac-Moody algebras. Journal of Algebra 1988; 115: 501-512.
2. Bennett C. Imaginary roots of Kac Moody algebra whose reflections preserve root multiplicities. Journal of Algebra 1993; 158: 244-267.
3. Casperson D. Strictly imaginary roots of Kac Moody algebras. Journal of Algebra 1994; 168: 90-122.
4. CHEN Hongji, LIU Bin. Super Hyperbolic Type Kac-Moody Lie Algebra. System Science and Mathematical sciences 1997; 10(4): 329-332.
5. Sthanumoorthy N, LillyPL. On the root systems of generalized Kac-Moody algebras. J.Madras University (WMY-2000 special issue) Section B:Sciences 2000; 52: 81-103.
6. Sthanumoorthy N, Lilly PL. Special imaginary roots of generalized Kac-Moody algebras. Communications in Algebra 2002; 30: 4771-4787
7. Sthanumoorthy N, Lilly PL. A note on purely imaginary roots of generalized Kac-Moody algebras. Communication in Algebra 2003; 31: 5467-5480.
8. Sthanumoorthy N,Lilly PL. On some classes of root systems of generalized Kac-Moody algebras. Contemp. Math. AMS. 2004; 343: 289-313.
9. Sthanumoorthy N,Lilly PL. Complete classifications of Generalized Kac-Moody algebras possessing special imaginary roots and strictly imaginary property. Communications in Algebra(USA) 2004; 35(8): 2450-2471.
10. Xinfang Song, Yinglin Guo. Root Multiplicity of a Special Generalized Kac- Moody Algebra EB_2. Mathematical Computation 2014; 3(3): 76-82.
11. Xinfang Song, Xiaoxi Wang Yinglin Guo. Root Structure of a Special generalized Kac-Moody algebras. Mathematical Computation2014; 3(3): 83-88.