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Eigen Values of non Symmetric Matrices

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ABSTRACT

Eigen value, or characteristic value, problems are a special case of boundary value problems that are common in engineering problems contexts involving vibrations, elasticity and other oscillating systems. A wide variety of methods are available for solving Eigen value problems. Most are based on two step process. The first step involves transforming the original matrix to simpler one that retains all the original Eigen values. Then iterative methods are used to determine these Eigen values. Jacobi method, Given's method and Householder method are used to find Eigen values of symmetric matrix. Aside from symmetric matrix, LR and QR method are used to find Eigen values of non symmetric matrices. This paper discusses the ccomparison of the speed and accuracy of these methods. The results of this study indicate that the QR algorithm is more successful method for finding the Eigen values of a real non symmetric matrix.

KEYWORDS: QR method, LR Method, Non-symmetric matrix.

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INTRODUCTION

Iterative methods for finding all the Eigen values of non symmetric matrices have appeared only recently. The LR algorithm, based on the successive triangular decomposed of a matrix. A sequence of similar matrices is generated whose limit is triangular. A modified procedure for the LR algorithm improves numerical stability. It uses a modified decomposition with interchanges. The QR algorithm makes use of unitary transformations instead of triangular decomposition. A variation of this algorithm involves a double-shift technique for combining complex conjugate shifts of origin while using Householder's method. It was developed for finding complex conjugate Eigen values of real matrices and is known as the double QR procedure. The LR and QR transformations are applicable to the calculation of general matrices, but a large number of computational operations are required. Therefore, it is desirable to use a matrix of condensed form. Both of these transformations preserve the form of a Heisenberg or almost triangular matrix. The preliminary reduction of a matrix to Heisenberg form can be accomplished in several ways. However, in this paper we will be concerned with two important methods to solve discussing problems.

MATERIALS AND METHODOLOGY

The LR algorithm for finding all the Eigen values of an arbitrary matrix is an interpretation of QD scheme. Its basis is the triangular decomposition of a matrix. The matrix is factorized into the product of a unit left triangular matrix L and a right triangular matrix R such that

$$LR = A \quad (1.1)$$

The LR method begins with the original matrix A

A sequence of matrices is then formed such that A_{k+1} is derived from A_k decomposing it into L_k and R_k and forming the product of these in reverse order.

$$A_k = L_k R_k \quad , \quad A_{k+1} = R_k L_k \quad (1.2)$$

for $k=1,2, \dots$

In this process, a series of similarity transformations are performed on the original matrix A_1 , each of which consists of pre multiplication by a matrix which eliminates the sub diagonal elements and post-multiplication by its inverse.

$$A_k = R_{\{k-1\}} L_{\{k-1\}} = L_{\{k-1\}}^{-1} A_{\{k-1\}} L_{\{k-1\}} = \\ L_{\{k-1\}}^{-1} \dots L_{\{1\}}^{-1} A_{\{1\}} A_{\{k-1\}} L_{\{1\}} L_{\{k-1\}}$$

(1.3)

Equation (1.3) can be rewritten as

$$L_1 L_2 \dots L_{\{k-1\}} A_k = A_1 L_1 L_2 \dots L_{\{k-1\}} \quad (1.4)$$

Using $A_k = L_k R_k$ and (1.4) it follows that

$$L_1 L_2 \dots L_{\{k-1\}} L_k R_k = A_1 L_1 L_2 \dots L_{\{k-1\}}$$

$$L_1 L_2 \dots L_{\{k-1\}} R_{\{k-1\}} = A_1 L_1 L_2 \dots L_{\{k-2\}}$$

And so on. If

$$T_k = L_1 L_2 \dots L_k \tag{1.5}$$

$$U_k = R_k R_{\{k-1\}} \dots R_1$$

Where T_k is unit left triangular and U_k is right triangular, then

$$\begin{aligned} T_k U_k &= L_1 L_2 \dots L_{\{k-1\}} (L_k R_k) R_{\{k-1\}} \dots R_1 \\ &= L_1 L_2 \dots R_{\{k-1\}} A_k \dots R_1 \\ &= A_1 L_1 L_2 \dots R_{\{k-1\}} L_{\{k-1\}} \dots R_1 \\ &= A_1^2 L_1 L_2 \dots R_{\{k-2\}} L_{\{k-2\}} \dots R_1 \\ &= A_1^k \end{aligned} \tag{1.6}$$

The triangular decomposition of A_1^k is $T_k U_k$. All of the matrices A_k have the same Eigen values since they are similar. Under certain conditions, A_k tends to a right triangular matrix as $k \rightarrow \infty$.

The diagonal elements of this right triangular matrix are the Eigen values of A_k appearing in decreasing order of magnitude from left to right. The problem is that of determining the matrices L and R of (1.1) directly from A without going through any intermediate steps. Therefore $(r-1)$ rows of L and R can be determined by equating the elements in the first $(r-1)$ rows of both sides of equation (1.1). The elements are determined in the following order: First row of R , second row of L ; second row of R and so forth. Another order in which they can be determined is the first row of R , first column of L ; second row of R , second column of L , and so forth.

The LR algorithm is not suitable for full matrices because of the high volume of computations. Thus, it is necessary to reduce the original matrix to some condensed form which is invariant with respect to the LR transformation. One such form is the upper Heisenberg matrix which is an almost triangular matrix with zeros in position (i,j) for $i > j + 1$. A full matrix may be reduced to Heisenberg form in a stable manner by the use of similarity transformations.

QR algorithm: The QR transformation uses a decomposition of matrix A into the product of a unitary matrix Q and a right triangular matrix R .

A sequence of matrix defined starting with $A=A_1$, such that

$$A_k = Q_k R_k, A_{[k+1]} = R_k Q_k \quad (2.1)$$

For $k=1, 2, \dots, n$.

A_{k+1} is formed by post multiplying R_k by Q_k . This algorithm can be written as a similarity transformation

$$\begin{aligned} A_k &= Q_{[k-1]} R_{[k-1]} = Q_{[k-1]}^{[H]} A_{[k-1]} Q_{[k-1]} \\ &= Q_{[k-1]}^{[-1]} \dots Q_{[1]}^{[-1]} A_1 A_{[k-1]} Q_1 \dots Q_{[k-1]} \end{aligned} \quad (2.2)$$

Above equation gives

$$Q_1 Q_2 \dots Q_{[k-1]} A_k = A_1 Q_1 Q_2 \dots Q_{[k-1]} \quad (2.3)$$

Using $A_k = Q_k R_k$ and (2.3), we have

$$Q_1 Q \dots Q_{[k-1]} Q R_k = A_1 Q_1 Q_2 \dots Q_{[k-1]}$$

$$Q_1 Q_2 \dots Q_{[k-1]} R_{[k-1]} = A_1 Q_1 Q_2 \dots Q_{[k-2]} \quad (2.4)$$

And so on.

If

$$P_k = Q_1 Q_2 \dots Q_k \quad (2.5)$$

$$S_k = R_k R_{[k-1]} \dots R_1$$

Where T_k is unitary and S_k is right triangular, then

$$\begin{aligned} P_k S_k &= Q_1 Q_2 \dots Q_{[k-1]} (Q_k R_k) R_{[k-1]} \dots R_1 \\ &= Q_1 Q_2 \dots R_{[k-1]} A_k \dots R_1 \\ &= A_1 Q_1 Q_2 \dots R_{[k-1]} Q \dots R_1 \\ &= A_1^2 Q_1 Q_2 \dots R_{[k-2]} Q_{[k-2]} \dots R_1 \\ &= A_1^k \end{aligned} \quad (2.6)$$

Francis proved that for any matrix A there exists a unitary matrix Q such that $A= QR$ where R is a right triangular matrix which has real non-negative, diagonal elements. Moreover, the Q is unique if A is non-singular. Thus the unitary-triangular decomposition of any square matrix exists, and, if the matrix is non-singular, the decomposition is unique.

Francis proved that if A is a non-singular matrix with all Eigen values of distinct modulus then, if $k \rightarrow \infty$, the elements of A_k below the diagonal tend to zero, and the elements on the

diagonal tend to the Eigen values of A. The factorization of a matrix A_k into Q_k and R_k involves the use of elementary unitary transformations

CONCLUSION

The results of this study indicate that the QR algorithm is a more successful method than the LR algorithm for finding the Eigen values of real non symmetric matrices. The convergence properties of the modified LR algorithm applied to real matrices; are not satisfactory except when the Eigen values of the matrix are all real. Another restriction is that for convergence to take place any matrix with equal Eigen values must have linear divisors. Also, few matrices which are ill-conditioned with respect to their Eigen values were found to become progressively more ill-conditioned at every iteration points. The total number of iterations in the convergence to the Eigen values must be taken into consideration when comparing the times required by the LR and QR algorithms. As the dimension of the test matrices used was increased, the number of iterations required for convergence by the LR increased at a faster rate than for the QR. When the total number of iterations is the same for both algorithms, the double QR requires more time. This is again due to the number of multiplications involved in one iteration point. The LR transformation is much easier to apply, but because of its possible numerical instability and other restrictions, its value seems limited. Therefore, the QR algorithm was found to be the more successful method for dealing with the real non symmetric Eigen value problem.

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REFERENCES:

1. Francis, JGF. QR Transformation- A Unitary Analogue to the LR Transformation - Part 1. The Computer Journal. 1961; 4: 265-271.
2. Francis, JGF. The QR Transformation- Part 2. The Computer Journal. 1961; 4: 332-345.
3. Henrici, P. The Quotient-Difference Algorithm. National Bureau of Standards Applied Mathematics Series. 1958; 49: 23-46.
4. Henrici, P. and Watkins, BO. Finding Zeros of a Polynominal by the Q-D Algorithm. Communications of the Association for Computing Machinery. 1965; 8: 570-574.

5. Householder, AS. On Deflating Matrices. *Journal of the Society of Industrial and Applied Mathematics*. 1961; 9: 89-93.
 6. Parlett, B. The Development and Use of Methods of LR. *Siam. Review*. 1964; 6: 275-295.
 7. Ralston, A. *A First Course in Numerical Analysis*, McGraw-Hill, Inc., New York, New York. 1965; 96-520.
 8. Rutishauser, H. Solution of Eigen value Problems With the LR Transformation. *National Bureau of Standards Applied Mathematics Series*. 1958; 49: 47-81.
 9. Wilkinson, JH. Stability of the Reduction of a Matrix to Almost Triangular and Triangular Form by Elementary Similarity Transformations. *Journal of the Association for computing Machinery*. 1959; 6: 336-359.
 10. Wilkinson, JH. householder Method for the Solution of the Algebraic Eigen problem. *The Computer Journal*. 1960; 3: 23-27.
 11. Wilkinson, JH. Convergence of the LR, QR, and Related Algorithms. *The Computer Journal*. 1965; 8: 77-84.
 12. Wilkinson, JH. *The Algebraic Eigen value Problem*. Oxford University Press; London. 1965; 264.
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