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Solution of the fractional partialdifferential equation by using homotopy analysisi method

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ABSTRACT:

The solution of fractional partial differential equations is obtained by using the homotopy analysis method. We also discussed the convergence analysis of the homotopy analysis method about the considered fractional partial differential equation.

KEY WORDS: Fractional partial differential equation, Convergence analysis, Homotopy analysis method.

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INTRODUCTION OF HOMOTOPY ANALYSIS METHOD:

This method is proposed by Liao in 1992[3-6]. The following differential equations are considered by us for this method,

$$N_i\left[u_i\left(x,t\right)\right]=0, i=1,2,\ldots,n \tag{1}$$

Where N_i considered as a nonlinear operators, (x,t) and $u_i(x,t)$ are pair of independent variables and unknown functions respectively.

The so-called zero-order deformation equations defined by

$$(1-q)L\left[\varphi_i\left(x,t,;q\right)-u_{i,0}\left(x,t\right)\right]=qc_iN_i\left[\phi_i\left(x,t,;q\right)\right] \tag{2}$$

Where q is an embedding parameter which lies between $[0,1]$, c_i and L are nonzero auxiliary functions and auxiliary linear operator respectively, initial guesses of $u_i(x,t)$ are $u_{i,0}(x,t)$ and $\phi_i(x,t,;q)$ are unknown functions.

We have freedom to choose auxiliary objects such as c_i and L in HAM, which is main importance of this method.

When $q=0$ *and* $q=1$ we get by (2),

$$\phi_i\left(x,t,;0\right)=u_{i,0}\left(x,t\right) \quad and \quad \phi_i\left(x,t,;1\right)=u_i\left(x,t\right)$$

By Taylor's series expansion

$$\phi_i\left(x,t,;q\right)=u_{i,0}\left(x,t\right)+\sum_{m=1}^{\infty}u_{i,m}\left(x,t\right).q^m \tag{3}$$

Where

$$u_{i,m}\left(x,t\right)=\left[\frac{1}{m!}.\frac{\partial^m\phi_i\left(x,t,;q\right)}{\partial q^m}\right]_{q=0} \tag{4}$$

If we choose the auxiliary parameter c_i ,the auxiliary functions, the auxiliary linear operator and initial guesses properly than the series equation (3) converges at $q=1$

$$\phi_i\left(x,t,;1\right) = u_{i,0}\left(x,t\right) + \sum_{m=1}^{\infty} u_{i,m}\left(x,t\right) \tag{5}$$

This must be one of the solutions of the original nonlinear equations.

If we differentiate equation (2) m times with respect to the embedding parameter q and the setting $q = 0$ and finally dividing them by $m!$ than we get the so-called m^{th}Order deformation equations like this

$$L\left[u_{i,m}\left(x,t\right) - \chi_m u_{i,m-1}\left(x,t\right)\right] = h_i R_{i,m}(\overrightarrow{u_{i,m-1)}}) \tag{6}$$

Where

$$R_{i,m}\left(u_{i,m-1)}\right) = \left[\frac{1}{(m-1)!}.\frac{\partial^{m-1} N_i[\phi_i\left(x,t,;q\right)}{\partial q^{m-1}}\right]_{q=0} \tag{7}$$

χ_m is characteristic function.

FRACTIONAL DERIVATIVE ACCORDING TO RIEMANN-LIOUVILLE:

$$D^{\alpha}\left(t^n\right) = \frac{d^{\alpha}}{dt^{\alpha}}\left(t^n\right) = \frac{\Gamma\left(n+1\right)}{\Gamma\left(n+1-\alpha\right)} t^{n-\alpha} \tag{8}$$

Where, Gamma possesses a standard definite. Inverse fractional differential operator notation is J^{α}

SOLUTION OF FRACTIONAL KDV EQUATION BY USING THE HOMOTOPY ANALYSIS METHOD: $[0 < \alpha \le 1]$

Let following be the KDV equation [9]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - 3\frac{\partial u^2}{\partial x} = 0 \tag{9}$$

With initial condition $u(x,0) = 3x$

First of all we want to define linear and nonlinear terms like as

$$N(\phi\left(x,t;q\right)) = \frac{\partial^{\alpha}\phi\left(x,t;q\right)}{\partial t^{\alpha}} - 3\frac{\partial\phi\left(x,t;q\right)^2}{\partial x} \tag{10}$$

$$L(\phi(x,t;q)) = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha} \qquad (11)$$

Assume initial approximation

$$u_0(x,t) = 6x + tx^3$$

By using the procedure of Homotopy Analysis Method, the zeroth-order deformation equations for (1) can be written as

$$(1-q)\left[\phi(x,t;q) - u_0(x,t)\right] = qc_0 N\left[\phi(x,t;q)\right] \qquad (12)$$

For $q = 0$ and $q = 1$, it can be written as

$$\phi(x,t;0) = u_0(x,t) \quad ,\phi(x,t;1) = u(x,t)$$

The mth order deformation equations can be written as

$$L\left(u_m(x,t) - \chi_m u_{m-1}(x,t)\right) = c_0 R_m(u_{m-1}) \qquad (13)$$

Where $R_m(u_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - 3\frac{\partial {u_{m-1}}^2}{\partial x}$ (14)

The approximate solution of equation (9) can be written as

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \qquad (15)$$

Where $u_m(x,t) = \chi_m u_{m-1}(x,t) + c_0 L^{-1}\left[R_m(u_{m-1})\right]$ (16)

If we take m=1 in (14),

$$R_1(u_0) = \frac{\partial^\alpha u_0}{\partial t^\alpha} - 3\frac{\partial {u_0}^2}{\partial x} = x^2\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{2-\alpha} - 54x - 54tx^2 - 12t^2x^3$$

$$\begin{aligned} u_1(x,t) &= c_0 D^{-\alpha}\left[R_1(u_0)\right] \\ &= c_0 D^{-\alpha}\left[x^2\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{2-\alpha} - 54x - 54tx^2 - 12t^2x^3\right] \\ &= c_0\left[\frac{x^2t^2}{2}x^3 - 54x\frac{1}{\Gamma(1+\alpha)}t^\alpha - 54x^2\frac{1}{\Gamma(2+\alpha)}t^{1+\alpha} - 12x^3\frac{1}{\Gamma(3+\alpha)}t^{2+\alpha}\right] \end{aligned} \qquad (17)$$

If we take two terms approximation than we get

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + \ldots\ldots \\ &= 3x + tx^2 + c_0\left[\frac{x^2t^2}{2}x^3 - 54x\frac{1}{\Gamma(1+\alpha)}t^\alpha - 54x^2\frac{1}{\Gamma(2+\alpha)}t^{1+\alpha} - 12x^3\frac{1}{\Gamma(3+\alpha)}t^{2+\alpha}\right] \end{aligned} \qquad (18)$$

If we take some special case,

$\alpha = 1$, $\alpha = 1$, $\quad u(x,t) = 3x + tx^2 + c_0\left[\frac{x^2t^2}{2}x^3 - 54xt - 27x^2t^2 - 2x^3t^3\right]$

$\alpha = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\quad u(x,t) = 3x + tx^2 + c_0\left[\frac{x^2t^2}{2}x^3 - 108x\frac{1}{\sqrt{\pi}}t^{\frac{1}{2}} - 36x^2\frac{1}{\sqrt{\pi}}t^{\frac{3}{2}} - \frac{16}{5}x^3\frac{1}{\sqrt{\pi}}t^{\frac{5}{2}}\right]$

CONVERGENCE OF HOMOTOPY ANALYSIS METHOD (HAM):

Theorem:-As long as the series equation (15) is convergent where $u_m(x,t)$ is governed by the mth order deformation equation (13) under (14) must be the solution of (9) [5].

Proof: Let the series $u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)$ be convergent.

Then

$$\lim_{m\to\infty} u_m(x,t) = 0 \tag{19}$$

Now we have

$$\begin{aligned}&\sum_{m=1}^{n}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right]\\ &= u_1 + (u_2 - u_3) + (u_3 - u_2) + \ldots + (u_{n-1} - u_{n-2}) + (u_n - u_{n-1})\\ &= u_n\end{aligned}$$

So

$$\sum_{m=1}^{\infty}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \lim_{n\to\infty} u_n(x,t) = 0 \tag{20}$$

According to the definition of linear operators, we can write

$$\sum_{m=1}^{\infty} L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = L\left(\sum_{m=1}^{\infty}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right]\right) = L(0) = 0$$

From the above equation and equation (13)

$$R_m\left(\overrightarrow{u_{m-1}}\right) = 0, \quad (\because c_0 \neq 0) \tag{21}$$

From (14),

$$\sum_{m=1}^{\infty} R_m\left(\overrightarrow{u_{m-1}}\right)$$
$$= \sum_{m=1}^{\infty} \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - \sum_{m=1}^{\infty} \frac{\partial\left(u_{m-1}\right)^2}{\partial x} \qquad (22)$$
$$= \frac{\partial^\alpha \sum_{m=1}^{\infty} u_{m-1}}{\partial t^\alpha} - 3\frac{\partial\left(\sum_{m=1}^{\infty} u_{m-1}\right)^2}{\partial x} + 3\frac{\partial\left(\sum_{i,j=1}^{\infty} u_i u_j\right)}{\partial x}$$

From (21) and (22), proof of the theorem is completed.

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