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Derivation of Kepler's Law of Planetary Motion from Newton's Law of Gravitation and Vice Versa

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ABSTRACT:

One of the greatest scientists cum mathematician Isaac Newton propounded the theory of Gravitational Force. This theory invited huge reaction throughout the world and scientists took it as a revolutionary movement in the field of science. The concept that every point mass attracts every other point mass by a force acting along the line intersecting both points and the force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them. This invented a mathematical equation. While Kepler's three laws of planetary motion can be described as - the path of the planets about the sun is elliptical in shape, with the center of the sun being located at one focus and an imaginary line drawn from the center of the sun to the center of the planet will sweep out equal areas in equal intervals of time. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the sun. (The Law of Harmonies)

The present paper is an endeavour to find a relation between the two reputed conceptual theories. It also aims at analyzing the two theories in mathematical context. It displays that Newton's law of gravitation follows from Kepler's law of planetary motion.

KEY WORDS: Gravitational force, planetary motion, derivation, mathematical equation.

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INTRODUCTION:

Newton:

Isaac Newton, "the supreme that the human race has produce- he who in genius surpassed the human kind"[1], was born on January 4, 1643, in Woolsthorpe, Lincolnshire, England. Fortunately, Newton was not a good farmer, and was sent to Cambridge to become a preacher. At Cambridge, Newton studied mathematics, being especially strongly influenced by Euclid, although he was also influenced by Baconian and Cartesian philosophies. Newton was forced to leave Cambridge when it was closed because of the plague, and it was during this period that he made some of his most significant discoveries. With the reticence he was to show later in life, Newton did not, however, publish his results.

Pondering why the apple never drops sideways or upwards or any other direction except perpendicular to the ground, Newton realized that the Earth itself must be responsible for the apple's downward motion. Theorizing that this force must be proportional to the masses of the two objects involved, and using previous intuition about the inverse-square relationship of the force between the earth and the moon, Newton was able to formulate a general physical law by induction.

DISCUSSION:

Newton's law of Universal Gravitation:

It states that a particle attracts every other particle in the universe with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. This is a general physical law derived from empirical observations by what Isaac Newton called inductive reasoning. It is a part of classical mechanics and was formulated in Newton's work *Philosophiæ Naturalis Principia Mathematica* ("the *Principia*"), first published on 5 July 1686. When Newton's book was presented in 1686 to the Royal Society, Robert Hooke made a claim that Newton had obtained the inverse square law from him.

In modern sense the law states that every point mass attracts every other point mass by a force acting along the line intersecting both points. The force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them.

The equation for universal gravitation thus takes the form:

$$F = G\frac{m_1 m_2}{r^2}$$

Where F is the gravitational force acting between two objects, m_1 and m_2 are the masses of the objects, r is the distance between the centers of their masses, and G is the gravitational constant.

Newton's law of gravitation resembles Coulomb's law of electrical forces, which is used to calculate the magnitude of the electrical force arising between two charged bodies. Both are inverse-square laws, where force is inversely proportional to the square of the distance between the bodies. Coulomb's law has the product of two charges in place of the product of the masses, and the electrostatic constantin place of the gravitational constant. Newton's law has since been superseded by Albert Einstein's theory of general relativity, but it continues to be used as an excellent approximation of the effects of gravity in most applications. Relativity is required only when there is a need for extreme precision, or when dealing with very strong gravitational fields, such as those found near extremely massive and dense objects, or at very close distances (such as Mercury's orbit around the Sun). Curtis Wilson terms, “Newton proposed taking as the forces of Saturn’s orbit the centre of gravity of Jupiter and the Sun and introduced Horrocksian- style oscillation into Saturn’s eccentricity and line of apsides…[2]

Johannes Kepler was influenced by Copernicus and delighted in his ideas. In 1596, while a mathematics teacher in Graz, he wrote the first outspoken defense of the Copernican system, the Mysterium Cosmographicum. In 1619 he published *Harmonices Mundi*, in which he describes his third law.

Kepler's Laws of Planetary Motion:

Kepler's three laws of planetary motion can be described as follows:

- The path of the planets about the sun is elliptical in shape, with the center of the sun being located at one focus. (The Law of Ellipses)
- An imaginary line drawn from the center of the sun to the center of the planet will sweep out equal areas in equal intervals of time. (The Law of Equal Areas)
- The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their average distances from the sun. (The Law of Harmonies)

The usefulness of Kepler’s laws extends to the motions of natural and artificial satellites as well as to unpowered spacecraft in orbit in stellar systems or near planets. As formulated by Kepler, the laws do not, of course, take into account the gravitational interactions (as perturbing effects) of the various planets on each other. The general problem of accurately predicting the motions of more than two bodies under their mutual attractions is quite complicated; analytical solutions of the three-body problem are unobtainable except for some special cases. It may be noted that Kepler's laws apply not only to gravitational but also to all other inverse-square-law forces and, if due allowance is made for relativistic and quantum effects, to the electromagnetic forces within the atom.

METHODS:

Derivation of Newton's law of planetary motion from Kepler's laws.

$$\text{According to Kepler's second law, areal velocity is constant}$$

$$\text{i.e. } \frac{h}{2} \text{ is constant.}$$

$$\text{Value of } h \text{ is } r^2\dot{\theta}$$

$$\frac{1}{2}\left(r^2\dot{\theta}\right) = \text{constant}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Thus transverse acceleration is zero and the planet has radial acceleration only directed towards the sun which shows that force of attraction is central and directed towards the sun.

Acccording to Kepler's first law, path of a planet is elliptic about the sun.

The equation of the ellipse having pole at the sun as focus is

$$\frac{l}{r} = 1 + e\cos\theta \text{ where } e < 1$$

$$ul = 1 + e\cos\theta$$

$$u = \frac{1}{l}\left[1 + e\cos\theta\right]$$

$$Differentiating\ we\ get$$

$$\frac{du}{d\theta}=\frac{1}{l}\left[-e\sin\theta\right]\quad and\ \frac{d^2u}{d\theta^2}=-\frac{e}{l}\cos\theta$$

$$The\ force\ per\ unit\ mass\ for\ the\ central\ orbit\ is$$

$$F=h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right]$$

$$=h^2u^2\left[\frac{1+e\cos\theta}{l}-\frac{e\cos\theta}{l}\right]$$

$$=h^2u^2\left[\frac{1}{l}\right]=\frac{h^2}{r^2}\left[\frac{1}{l}\right]$$

$$T=The\ time\ period\ for\ an\ elliptic\ orbit$$

$$=\frac{Area\ of\ ellipse}{Rate\ of\ description\ of\ the\ sectorial\ area}=\frac{\pi ab}{h/2}$$

$$According\ to\ Kepler's\ third\ law$$

$$T^2\infty(2a)^3$$

$$\frac{T^2}{a^3}=constant$$

$$\frac{4\pi^2a^2b^2}{h^2a^3}=constant$$

$$\frac{4\pi^2}{h^2}\left(\frac{b^2}{a}\right)=constant$$

$$\frac{4\pi^2}{h^2}(l)=constant$$

$$\frac{l}{h^2}=constant\ for\ all\ planets$$

$$\frac{h^2}{l}=constant=\mu$$

$$F=\frac{1}{r^2}\left(h^2/l\right)=\frac{1}{r^2}(\mu)$$

$$\therefore\quad mF=\frac{m\mu}{r^2}$$

Thus the force of attraction between the sun and the planet of mass m varies inversely as the square of the distance between them.

This shows that Newton's law of gravitation follows from Kepler's law of planetary motion.

Derivation of Kepler law from Newton's law of gravitation.

If we assume that the sun is a fixed point and planets are attracted by the sun only, then the Kepler's law can be deduced from the Newton's law of Gravitation.

Let us take the sun as a fixed point O. Let P(r,θ) be the position of a planet at time *t*. By Newton's law of gravitation, the force F per unit mass of the planet is given by

$$F = \frac{\mu}{r^2} \text{ where } \mu \text{ is a positive constant}$$

The differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{h^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0$$

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0$$

$$\frac{d^2x}{dt^2} + x = 0 \quad \text{Where } x = u - \frac{\mu}{h^2}$$

$$(D^2+1)x=0$$

Auxiliary equation is $D^2+1=0$

$$D^2=-1$$

$$D^2=\pm i$$

Thus the general solution is

$$x=c_1\cos(\theta-c_2)$$

Putting $x=u-\dfrac{\mu}{h^2}$

$$u-\frac{\mu}{h^2}=c_1\cos(\theta-c_2)$$

$$\frac{1}{r}-\frac{\mu}{h^2}=c_1\cos(\theta-c_2)$$

Dividing both sides by $\dfrac{\mu}{h^2}$, *we get*

$$\frac{h^2/\mu}{r}=1+\frac{c_1h^2}{\mu}\cos(\theta-c_2)$$ *which is the polar equation of the form*

$$\frac{l}{r}=1+e\cos\theta$$

This equation represents a with the conic with sun at one focus. The semi latus rectum is h^2/μ *and eccentricity is* $\dfrac{c_1h^2}{\mu}$.

Since ellipse is the only closed conic, therefore the planet describes an ellipse with sun at one focus.

Thus Kepler's first law is derived from Newton's law.

Derivation of Kepler's second law.

Since the rate of description of sectorial area traced out by the radius vector joining the partical to a fixed point is constant and is equal to $\dfrac{h}{2}$ *for the central orbit.*

Hence Kepler's second law is verified.

$$\text{Derivation of Kepler's third law.}$$

$$\text{If } T \text{ is the time of one revolution along the orbit.}$$

$$\text{Then } T = \frac{\text{Area of the ellipse}}{\text{Rate of description of the sectorial area}}$$

$$= \frac{\pi ab}{h/2}$$

$$= \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\frac{\mu b^2}{a}}}$$

$$= \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$T^2 = \frac{\pi^2}{2\mu}(2a)^3$$

$$\text{Hence Kepler's third law is verified.}$$

Thus Newton's open new avenues of knowledge, "Newton demonstrated the scope of the "new analysis" by using power series and fluxions to solve problems concerning optimization, tangents, and curvature, as well as the area, length and centre of gravity of curves…"[3]

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