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Study of Inter Relation Between Two Types of Continuous Functions On Convex Topological Space

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ABSTRACT

The concept of $\delta - \mathcal{C}$ continuity and $\delta_{\star} - \mathcal{C}$ continuity already have been introduced earlier on convex topological space $(X , \tau , \mathcal{C})$ where τ is the topology and $\mathcal{C}$ is the convexity on the same underlying set X . In this paper I have mainly investigated the inter relation between these two types of continuity .

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KEYWORDS : Convex topological space , $\delta - \mathcal{C}$ continuous function , $\delta_{\star} - \mathcal{C}$ continuous function .

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1. INTRODUCTION

The development of "abstract convexity" has emanated from different sources in different ways ; the first type of development basically banked on generalization of particular problems such as separation of convex *sets*[1], *extremality*[2,3] or continuous *selection*[4] . The second type of development lay before the reader such axiomatizations , which in every case of design , express particular point of view of convexity . With the view point of generalized topology which enters into convexity via the closure or hull operator , Schmidt and Hammer, , introduced some axioms to explain abstract convexity . The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birch off and Frink , Schmidt, Hammer .

The axiomatizations as proposed by M.L.J. Van De Vel in his paper Theory of Convex *Structure*[5] will be followed through out in this paper .

The author has discussed in " Topology and Convexity on the same *set*[6] " and introduced the compatibility of the topology with a convexity on the same underlying set . At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet $(X, \tau, \mathcal{C})$ and call it convex topological space only to bring back "compatibility" in another way subsequently . With this compatibility , Van De Vel has called the triplet $(X, \tau, \mathcal{C})$ a topological convex structure

In this paper , Art. 2 deals with some early definitions , results and in Art. 3 I have discussed mainly inter relation between $\delta - \mathcal{C}$ continuous function and $\delta_\star - \mathcal{C}$ continuous function .

2. PREREQUISITES :

Definition 2.1[6] : Let X be a non empty set . A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\phi, X \in \mathcal{C}$
2. $\mathcal{C}$ is stable for intersection , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty , then $\cap \mathcal{D} \in \mathcal{C}$
3. $\mathcal{C}$ is stable for nested unions , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty and totally ordered by set inclusion, then $\cup \mathcal{D} \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure . The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets .

Definition 2.2[6] : Let $\mathcal{C}$ be a convexity on set X . Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by $co(A) = \cap \{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3[6] : Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X . The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y; called the relative convexity of Y.

Note 2.4[6] : The hull operator co_Y of a subspace $(Y, \mathcal{C}_Y)$ satisfy the following :

$\forall A \subseteq Y : co_Y(A) = co(A) \cap Y$.

Definition 2.5 [6]: Let $(X, \mathcal{C})$ be a convex structure and let τ be a topology on X . Then τ is said to be compatible with the convex structure $(X, \mathcal{C})$ if all polytopes of $\mathcal{C}$ are closed in τ where polytopes means convex hull of a finite set . Also the triplet $(X, \tau, \mathcal{C})$ is then called topological convex structure .

Note 2.6 [6] : Let $(X, \tau, \mathcal{C})$ be a topological convex structure . Then collection of all closed sets in (X, τ) are subset of $\mathcal{C}$.

Definition 2.7 [7] : Let (X, τ) be a topological space and let $\mathcal{C}$ be a convexity on X . Then the triplet $(X, \tau, \mathcal{C})$ is called a convex topological space (CTS in short) .

Theorem 2.8 [7] : Let $(X, \tau, \mathcal{C})$ be a convex topological space . Let A be a subset of X . Consider the set A_* , where A_* is defined as follows : $A_* = \{x \in X : co(U) \cap A \neq \emptyset, x \in U \in \tau\}$. Then the collection $\tau_* = \{A^c : A \subseteq X, A = A_*\}$ is a topology on X such that $\tau_* \subseteq \tau$.

Definition 2.9 [8] : Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is said to be $\delta - \mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f(int(U_*)) \subseteq int(V_*)$.

Definition 2.10 [9] : Let $(X, \tau, \mathcal{C}_1)$ and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is said to be $\delta_\star - \mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f\left(int(co(U))\right) \subseteq int(co(V))$.

Definition 2.11 [10] : A convex topological space $(X, \tau, \mathcal{C})$ is said to be an $S\mathcal{C} - R$ space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int(U_*) \subseteq V$.

Definition 2.12 [11] : A convex topological space $(X, \tau, \mathcal{C})$ is said to be a semi $\mathcal{C}$- regular space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int(co(U)) \subseteq V$.

3. COMPARISON BETWEEN $\delta - \mathcal{C}$ CONTINUOUS AND $\delta_\star - \mathcal{C}$ CONTINUOUS FUNCTIONS :

Already I have discussed detail the concept of $\delta - \mathcal{C}$ continuity[8] and $\delta_\star - \mathcal{C}$ continuity[9] on convex topological space . Now I will show that these two concepts are independent in general which follow from the next two examples .

Example 3.1 : Let us consider the function $f:(X,\tau,\mathcal{C}_1)\rightarrow(X,\sigma,\mathcal{C}_2)$ where $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$, $\mathcal{C}_1=\{\emptyset,X\}$, $\sigma=\{\emptyset,X,\{a\}\}$, $\mathcal{C}_2=\{\emptyset,X,\{a,b\}\}$ and f is the identity mapping I_X on X .

In the convex topological space $(X,\sigma,\mathcal{C}_2)$, we see that $\{a\}_\star=\{b\}_\star=\{c\}_\star=X$. This shows that the function f is $\delta-\mathcal{C}$ continuous .

Again for the point a in $(X,\tau,\mathcal{C}_1)$ we consider the open nbd. $V=\{a\}$ of $f(a)=a$ in $(X,\sigma,\mathcal{C}_2)$. In the CTS $(X,\sigma,\mathcal{C}_2)$, $co(\{a\})=\{a,b\}$ and $int\big(co(\{a\})\big)=\{a\}$. Also in the CTS $(X,\tau,\mathcal{C}_1)$, $(\{a\})=co(\{a,b\})=co(X)=X$. Thus there is no open nbd. $U\in\tau$ of a such that $f\Big(int\big(co(U)\big)\Big)\subseteq\Big(int\big(co(V)\big)\Big)$. This shows that f is not $\delta_\star-\mathcal{C}$ continuous .

Hence we conclude that $\delta-\mathcal{C}\ continuity\nRightarrow\delta_\star-\mathcal{C}\ continuity$.

Example 3.2 : Let us consider the function $f:(X,\tau,\mathcal{C}_1)\rightarrow(X,\sigma,\mathcal{C}_2)$ where $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{b\}\}$, $\mathcal{C}_1=\{\emptyset,X,\{b\}\}$, $\sigma=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$, $\mathcal{C}_2=\{\emptyset,X,\{b\}\}$ and f is the identity mapping I_X on X .

In the CTS $(X,\tau,\mathcal{C}_1)$, non-empty open sets are $\{b\}$, X and $int(co(\{b\})=\{b\}$, $\big(co(X)\big)=X$. Also in the CTS $(X,\sigma,\mathcal{C}_2)$, non-empty open sets are $\{a\}$, $\{b\}$, $\{a,b\}$, X and $int(co(\{a\})=int\big(co(\{a,b\})\big)=int\big(co(X)\big)=X$, $int\big(co(\{b\})\big)=\{b\}$. So we see that for each $x\in X$ in $(X,\tau,\mathcal{C}_1)$ and each open nbd. V of $f(x)$ in $(X,\sigma,\mathcal{C}_2)$, there exists an open nbd. U of x such that $f\Big(int\big(co(U)\big)\Big)\subseteq int\big(co(V)\big)$. This shows that the function f is $\delta_\star-\mathcal{C}$ continuous .

Again for the point a in $(X,\tau,\mathcal{C}_1)$ we consider the open nbd. $V=\{a\}$ of $f(a)=a$ in $(X,\sigma,\mathcal{C}_2)$. Now in the CTS $(X,\sigma,\mathcal{C}_2)$, $(\{a\}_\star)=int(\{a,c\})=\{a\}$. Also in the CTS $(X,\tau,\mathcal{C}_1)$, open nbd. of a is X and $int(X_\star)=X$. Thus there is no open nbd. $U\in\tau$ of a such that $f\big(int(U_\star)\big)\subseteq int(V_\star)$. This shows that f is not $\delta-\mathcal{C}$ continuous .

Hence we conclude that $\delta_\star-\mathcal{C}\ continuity\nRightarrow\delta-\mathcal{C}\ continuity$.

Now I will discuss under what conditions these two concepts coincide .

Theorem 3.3 : If a function $f:(X,\tau,\mathcal{C}_1)\rightarrow(Y,\sigma,\mathcal{C}_2)$ be $\delta-\mathcal{C}$ continuous and Y is an $S\mathcal{C}-R$ space where τ is compatible with $\mathcal{C}_1$, then f is $\delta_\star-\mathcal{C}$ continuous function .

Proof : Since τ is compatible with $\mathcal{C}_1$, all closed sets of X are in $\mathcal{C}_1$. Then for any $P\subseteq X$, $co(P)\subseteq\bar{P}\subseteq P_\star$ and so $int\big(co(P)\big)\subseteq int(P_\star)$. Let $x\in X$ and V be any open nbd. of $f(x)$. Since Y is an $S\mathcal{C}-R$ space , there exists an open set W such that $f(x)\in W\subseteq int(W_\star)\subseteq V$.

Again f is $\delta - \mathcal{C}$ continuous . So there exists an open nbd. U of x such that $f(int(U_\star)) \subseteq int(W_\star)$. Now $int(W_\star) \subseteq V \subseteq co(V) \Rightarrow int(W_\star) \subseteq int(co(V))$. This shows that $f\left(int(co(U))\right) \subseteq f(int(U_\star)) \subseteq int(W_\star) \subseteq int(co(V))$. Hence f is $\delta_\star - \mathcal{C}$ continuous function .

Theorem 3.4 : If a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be $\delta_\star - \mathcal{C}$ continuous and X is an $S\mathcal{C} - R$ space where σ is compatible with $\mathcal{C}_2$, then f is $\delta - \mathcal{C}$ continuous function .

Proof : Given σ is compatible with $\mathcal{C}_2$. Then for any $P \subseteq Y$, $co(P) \subseteq \bar{P} \subseteq P_\star$ and so $int(co(P)) \subseteq int(P_\star)$. Let $x \in X$ and V be any open nbd. of $f(x)$. Since f is $\delta_\star - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f\left(int(co(U))\right) \subseteq int(co(V))$. So $f\left(int(co(U))\right) \subseteq int(co(V)) \subseteq int(V_\star)$. Again X is an $S\mathcal{C} - R$ space . So there exists an open set W such that $x \in W \subseteq int(W_\star) \subseteq U$. Now $int(W_\star) \subseteq U \subseteq co(U) \Rightarrow int(W_\star) \subseteq int(co(U))$. This shows that $f(int(W_\star)) \subseteq f\left(int(co(U))\right) \subseteq int(V_\star)$. Hence f is $\delta - \mathcal{C}$ continuous function .

Theorem 3.5 : If a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be $\delta - \mathcal{C}$ continuous where X is a semi $\mathcal{C}$- regular space and Y is an $S\mathcal{C} - R$ space , then f is $\delta_\star - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Since Y is an $S\mathcal{C} - R$ space , there exists an open set W of $f(x)$ such that $f(x) \in W \subseteq int(W_\star) \subseteq V$. So $int(W_\star) \subseteq co(V) \subseteq int(co(V))$. Again f is $\delta - \mathcal{C}$ continuous . So there exists an open nbd. Z of x such that $f(int(Z_\star)) \subseteq int(W_\star)$. Also X is a semi $\mathcal{C}$- regular space . Thus there exists an open set U such that $x \in U \subseteq int(co(U)) \subseteq int(Z_\star)$. So $f\left(int(co(U))\right) \subseteq f(int(Z_\star)) \subseteq int(W_\star) \subseteq int(co(V))$. This shows that f is $\delta_\star - \mathcal{C}$ continuous function .

Theorem 3.6 : If a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be $\delta_\star - \mathcal{C}$ continuous where X is an $S\mathcal{C} - R$ space and Y is a semi $\mathcal{C}$- regular space , then f is $\delta - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int(V_\star)$ is an open nbd. of $f(x)$ and Y is an semi $\mathcal{C}$- regular space .So there exists an open set W of $f(x)$ such that $f(x) \in W \subseteq int(co(W)) \subseteq int(V_\star)$. Again f is $\delta_\star - \mathcal{C}$ continuous . So there exists an open nbd. Z of x such that $f\left(int(co(Z))\right) \subseteq int(co(W))$. Also X is a $S\mathcal{C} - R$ space . Thus there exists an open set U such that $x \in U \subseteq int(U_\star) \subseteq int(co(Z))$. So $f(int(U_\star)) \subseteq f\left(int(co(Z))\right) \subseteq int(co(W)) \subseteq int(V_\star)$. This shows that f is $\delta - \mathcal{C}$ continuous function .

Theorem 3.7 : If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be continuous and X is an $S\mathcal{C} - R$ space , then f is $\delta - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int(V_\star)$ is an open nbd. of $f(x)$. Since f is continuous , there exists an open nbd. W of x such that $f(W) \subseteq int(V_\star)$. Again X is an $S\mathcal{C} - R$ space . So there exists an open set $U \in \tau$ such that $\in U \subseteq int(U_\star) \subseteq W$. Thus $f\big(int(U_\star)\big) \subseteq f(W) \subseteq int(V_\star)$. This shows that f is $\delta - \mathcal{C}$ continuous function .

Theorem 3.8 : If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be continuous and X is a semi $\mathcal{C}$-regular space , then f is $\delta_\star - \mathcal{C}$ continuous function .

Proof : Let $x \in X$ and V be any open nbd. of $f(x)$. Now $int\big(co(V)\big)$ is an open nbd. of $f(x)$. Since f is continuous , there exists an open nbd. W of x such that $f(W) \subseteq int\big(co(V)\big)$. Again X is a semi $\mathcal{C}$- regular space . So there exists an open set $U \in \tau$ such that $x \in U \subseteq int\big(co(U)\big) \subseteq W$. Thus $f\Big(int\big(co(U)\big)\Big) \subseteq f(W) \subseteq int\big(co(V)\big)$. This shows that f is $\delta_\star - \mathcal{C}$ continuous function .

Theorem 3.9[10] : If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be $\delta - \mathcal{C}$ continuous and Y is an $S\mathcal{C} - R$ space , then f is continuous function .

Theorem 3.10[11] : If a function $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be $\delta_\star - \mathcal{C}$ continuous and Y is a semi $\mathcal{C}$- regular space , then f is continuous function .

Theorem 3.11 : Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a function where X , Y are $S\mathcal{C} - R$ spaces . Then f is continuous iff it is $\delta - \mathcal{C}$ continuous .

Proof : Follows from the Theorems 3.7 and 3.9 .

Theorem 3.12 : Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a function where X , Y are semi $\mathcal{C}$-regular spaces . Then f is continuous iff it is $\delta_\star - \mathcal{C}$ continuous .

Proof : Follows from the Theorems 3.8 and 3.10 .

Theorem 3.13 : Let $f : (X, \tau, \mathcal{C}_1) \to (Y, \sigma, \mathcal{C}_2)$ be a function where X , Y are $S\mathcal{C} - R$ and semi $\mathcal{C}$- regular spaces . Then f is $\delta - \mathcal{C}$ continuous iff it is $\delta_\star - \mathcal{C}$ continuous .

Proof: Follows from the Theorems 3.11 and 3.12 .

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