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Product Summability $(E,1)(N,P_n)$ of Conjugate Series Of Fourier Series

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ABSTRACT

In the present study, some results on the product summability $(E,1)(N,P_n)$ of Conjugate Fourier series have been established.

KEYWORDS: (E,q) summability, (N,P_n) summability, $(E,1)(N,P_n)$ summability.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 42A24, 42A20 and 42B08

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INTRODUCTION

The study of Nörlund (N, P_n) suumability of Fourier series and its allied series was first studied by Mears[1] and then afterwards so many results deduced on the product summability of Nörlund means by a regular summability (i.e. in the form of $X(N, P_n)$ or $(N, P_n)X$, where X is any regular summability). In the same context, Lal and Nigam[2], Lal and Singh[3], Prasad[4], Sahney[5], Sinha and Shrivastava[6] and many researchers gave interesting results under different criteria & conditions. Therefore by inspiring this, under a very general condition, we have established some results on $(E,1)(N, P_n)$ summability of conjugate series of Fourier series. As a result, we see that the product operator gives better approximated value than individual linear operator.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of its partial sums $\{S_n\}$. Let $\{p_n\}$ be any sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n$$

$$P_{-1} = p_{-1} = 0$$

Therefore,

The sequence-to-sequence transformation is given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$, as generated by the sequence of coefficients $\{p_n\}$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be (N, P_n) summable to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s.

The necessary and sufficient condition for the regularity of (N, P_n) method is

$$\frac{p_n}{P_n} \to 0, \qquad \text{as } n \to \infty$$

Let,

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k$$

If $E_n^1 \to s$, as $n \to \infty$ then $\sum_{n=0}^{\infty} a_n$ is said to be summable s by Euler means. Hardy[7]

On superimposing $(E, 1)$ transform on (N, P_n) transform, we have the product $(E, 1)(N, P_n)$ transform t_n^{EN} of the n^{th} partial series S_n of the series $\sum_{n=0}^{\infty} a_n$ which is given by

$$t_n^{EN} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{p_k}\sum_{v=0}^{k} p_{k-v}s_v\right\}$$

then, the infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $(E,1)(N,P_n)$ summable to the sum s,

if $t_n^{EN} \to s$ as $n \to \infty$ i.e. the limit exist.

Let, $f(t)$ be a periodic function with period 2π and Lebesgue-integrable over the interval $(-\pi,\pi)$. Then the Fourier series associated with f at any point t is defined by

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \qquad (1.1)$$

Then the conjugate series of (1.1) is

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \qquad (1.2)$$

We use the following notations throughout this paper

$$\psi(t) = \frac{1}{2}[f(x+t) - f(x-t)]$$

and

$$\widetilde{K}_n(t) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\} \qquad (1.3)$$

KNOWN RESULTS

Recently, Sinha and Shrivastava[6] have discussed the almost $(E,q)(N,P_n)$ summability of Fourier Series by proving the following

Theorem A. If f is a 2π periodic function of class $L^a ip\ \alpha$ then the degree of approximation by the product $(E,q)(N,P_n)$ summability mean on its Fourier series (1.1) is given by

$$\|\tau_n - f\|_\infty = o\left(\frac{1}{(n+1)^\alpha}\right) \quad 0 < \alpha < 1 \qquad (2.1)$$

where, τ_n is defined as

$$\tau_n = \frac{1}{(1+q)^n}\sum_{m=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}S_\nu\right\}$$

Further, Prabhakar and Saxena[8], have obtained an analogous result by generalised theorem A for $(E,1)(N,P_n)$ summability of Fourier series under different condition and criteria. The Theorems are as follows

Theorem B. Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$$C_n = \sum_{\nu=1}^{n} c_\nu \to \infty, \text{as } n \to \infty$$

If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right) C_\tau}\right] \text{ as } t \to +0 \qquad (2.2)$$

where, $\alpha(t)$ is a positive, monotonic and non-increasing function of t and $\log(n+1) = O[\{\alpha(n+1)\}C_{n+1}]$, as $n \to \infty$ (2.3)

then the Fourier series (1.1) is $(E,1)(N,P_n)$ summable to zero at point x.

MAIN RESULT

With this point of view, we here prove the following theorems.

Theorem 1. Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$$C_n = \sum_{\nu=0}^{n} c_\nu \to \infty \text{ as } n \to \infty$$

If

$$\Psi(t) = \int_0^t |\psi(u)| du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right) C_\tau}\right] \text{ as } t \to +0 \qquad (3.1)$$

where, $\alpha(t)$ is a positive, monotonic and non-increasing function of t and

$$\log(n+1) = O[\{\alpha(n+1)\}C_{n+1}], \text{ as } n \to \infty \qquad (3.2)$$

then the conjugate Fourier series (1.2) is $(E,1)(N,P_n)$ summable to

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\cot\left(\frac{t}{2}\right) dt$$

at every pt, where this integral exists.

Theorem 2: Let $\{c_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$C_n = \sum_{\nu=0}^{n} c_\nu \to \infty \text{ as } n \to \infty$$

If

$$\Psi(t) = \int_0^t |\psi(u)|du = o\left[\frac{t}{\log\left(\frac{1}{t}\right)}\right], \quad \text{as } t \to +0 \qquad (3.3)$$

then the conjugate Fourier series (1.2) is $(E,1)(N,P_n)$ summable to

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\cot\left(\frac{t}{2}\right)dt$$

at every pt, where this integral exists.

To prove the following Theorems, we require the following lemmas.

LEMMAS

Lemma 4.1

For $0 \leq t \leq \frac{1}{n+1}$, $|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$

Proof.

$$|\tilde{K}_n(t)| = \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}\pi}\left[\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}\right\}\right]$$

$$\leq \frac{1}{2^{n+1}t}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}\right]$$

$$= \frac{(2n+1)}{2^{n+1}t}.2^n$$

$$= O\left(\frac{1}{t}\right)$$

This completes the proof of Lemma 3.1

Lemma 4.2

For $\frac{1}{n+1} \leq t \leq \pi$, $|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$

Proof.

$$|\widetilde{K}_n(t)| = \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right|\left|e^{i\frac{t}{2}}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right| + \frac{1}{2^{n+1}t}\left|\sum_{k=\tau}^{n}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right|$$

$$= |K_1| + |K_2|$$

$$|K_1| \leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}\right\}\right|\left|e^{i\nu t}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\right|$$

$$= O\left(\frac{1}{t}\right)$$

Now considering second term and using Abel's lemma

$$|K_2| \leq \frac{1}{2^{n+1}t}\left|\sum_{k=\tau}^{n}\binom{n}{k}\mathrm{Re}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right\}\right|$$

$$\leq \frac{1}{2^{n+1}t}\sum_{k=\tau}^{n}\binom{n}{k}\frac{1}{P_k}\max_{0\leq m\leq k}\left|\sum_{\nu=0}^{k}p_{k-\nu}e^{i\nu t}\right|$$

$$= O\left(\frac{1}{t}\right)$$

This completes the proof of Lemma 3.2 Similarly,

Lemma 4.3

For $0 \leq t \leq \frac{1}{n}$,

$$|\widetilde{K}_n(t)| = O\left(\frac{1}{t}\right)$$

Lemma 4.4

For $\frac{1}{n} \leq t \leq \pi$,

$$|\widetilde{K}_n(t)| = O\left(\frac{1}{t}\right)$$

PROOF

Proof of Theorem 1:

Let, $\tilde{s}_n$ denote the partial sum of conjugate Fourier series (1.2) then following Zygmund, we have

$$\tilde{s}_n - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore the $(E,1)(N,P_n)$ transform of $\tilde{s}_n(x)$ is given by

$$\tilde{t}_n^{EN} - \tilde{f}(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi} \psi(t)\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{\nu=0}^{k} p_{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

$$= \int_0^{\pi} \psi(t)|\widetilde{K}_n(t)|\,dt$$

For $0 < \delta < \pi$, we have

$$\int_0^{\pi} \psi(t)\widetilde{K}_n(t)dt = \int_0^{1/n+1} \psi(t)\widetilde{K}_n(t)dt + \int_{1/n+1}^{\delta} \psi(t)\widetilde{K}_n(t)dt + \int_{\delta}^{\pi} \psi(t)\widetilde{K}_n(t)dt$$

$$= I_1 + I_2 + I_3 \quad \text{(say)} \qquad (5.1)$$

Now, by applying (3.1), (3.2) and (4.1), we have

$$|I_1| \leq \int_0^{1/n+1} |\psi(t)||\widetilde{K}_n(t)|dt$$

$$= O\int_0^{1/n+1} \frac{1}{t}|\psi(t)|dt$$

$$= O(n+1)\int_0^{1/n+1} |\psi(t)|dt$$

$$= O(n+1)\left[o\left\{\frac{1}{(n+1)\alpha(n+1)C_{n+1}}\right\}\right]$$

$$= o\left\{\frac{1}{\log(n+1)}\right\}$$

$$= o(1), \qquad \text{as } n \to \infty \qquad (5.2)$$

From condition (3.1), (3.2) and (4.2), we have

$$|I_2| \le \int_{1/_{n+1}}^{\delta} |\psi(t)||\widetilde{K}_n(t)|dt$$

$$= O\left[\int_{1/_{n+1}}^{\delta} |\psi(t)|\left(\frac{1}{t}\right)dt\right]$$

$$= O\left[\left\{\frac{1}{t}\psi(t)\right\}_{1/_{n+1}}^{\delta} + \int_{1/_{n+1}}^{\delta} \frac{1}{t^2}\psi(t)dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right)C_\tau}\right\}_{1/_{n+1}}^{\delta} + \int_{1/_{n+1}}^{\delta} o\left\{\frac{1}{t\alpha\left(\frac{1}{t}\right)C_\tau}\right\}dt\right]$$

Putting $\frac{1}{t} = u$ in second term

$$= O\left[o\left\{\frac{1}{\alpha(n+1)C_{n+1}}\right\} + \int_{1/_{\delta}}^{n+1} o\left\{\frac{1}{u\alpha(u)C_u}\right\}du\right]$$

$$= o\left\{\frac{1}{\log(n+1)}\right\} + o\left\{\frac{1}{\log(n+1)}\right\}$$

$$= o(1) + o(1), \quad \text{as } n \to \infty$$

$$= o(1), \quad \text{as } n \to \infty \tag{5.3}$$

By Riemann-Lebesgue lemma & by regularity condition of the method of summability, we have

$$|I_3| \le \int_{\delta}^{\pi} |\psi(t)||\widetilde{K}_n(t)|dt$$

$$= o(1), \quad \text{as } n \to \infty \tag{5.4}$$

Combining (5.1), (5.2), (5.3) and (5.4), we have

$$I_1 + I_2 + I_3 = o(1)$$

Hence we proved that

$$\tilde{t}_n^{(E,1)(N,P_n)} - \tilde{f}(x) = o(1), \quad \text{as } n \to \infty$$

This completes the proof of Theorem 1.

Proof of Theorem 2:

For $0 < \delta < \pi$,

$$\tilde{t}_n^{(E,1)(N,P_n)} - \tilde{f}(x) = \int_0^{\pi} \psi(t)\widetilde{K}_n(t)\,dt$$

$$= \int_0^{1/n} \psi(t)\widetilde{K}_n(t)dt + \int_{1/n}^{\delta} \psi(t)\widetilde{K}_n(t)dt + \int_{\delta}^{\pi} \psi(t)\widetilde{K}_n(t)dt$$

$$= J_1 + J_2 + J_3 \quad \text{(say)} \qquad (5.5)$$

On applying (3.3) and (4.3), we have

$$|J_1| = \int_0^{1/n} |\psi(t)|\left|\widetilde{K}_n(t)\right|dt$$

$$= O\left[\int_0^{1/n} \frac{1}{t}|\psi(t)|dt\right]$$

$$= O(n)\left[\int_0^{1/n}|\psi(t)|dt\right]$$

$$= O\left\{\frac{1}{\log(n)}\right\}$$

$$= o(1)\,, \text{ as } n \to \infty \qquad (5.6)$$

From (3.3) and (4.4), we have

$$|J_2| = \int_{1/n}^{\delta} |\psi(t)|\left|\widetilde{K}_n(t)\right|dt$$

$$= O\left[\int_{1/n}^{\delta} \frac{1}{t}|\psi(t)|dt\right]$$

$$= O\left[\left\{\frac{1}{t}\psi(t)\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{1}{t^2}\psi(t)dt\right]$$

$$= O\left[o\left\{\frac{1}{\log\left(\frac{1}{t}\right)}\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} o\left\{\frac{1}{t\log\left(\frac{1}{t}\right)}\right\}dt\right]$$

$$= o\left\{\frac{1}{\log(n)}\right\} + o(1)\left\{-\log\log\left(\frac{1}{t}\right)\right\}_{1/n}^{\delta}$$

$$= o(1) + o(1)\,, \quad \text{as } n \to \infty$$

$$= o(1)\,, \text{ as } n \to \infty \qquad (5.7)$$

Finally,

By using Riemann-Lebesgue theorem and regularity condition of summability, we have

$$|J_3| = \int_\delta^\pi |\psi(t)||\tilde{K}_n(t)|dt = o(1), \quad \text{as } n \to \infty \tag{5.8}$$

Combining (5.5), (5.6), (5.7) and (5.8) we have

$$\tilde{t}_n^{(E,1)(N,P_n)} - \tilde{f}(x) = o(1), \qquad \text{as } n \to \infty$$

This completes the proof of Theorem 2.

CONCLUSION

Several results concerning the product summability of Nörlund-Euler means have been reviewed with different criteria and conditions. In future, by applying more conditions we can rectify the errors and its application in the field of Fourier analysis.

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