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Some common fixed point theorems for three mappings inVector b-metric spaces

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ABSTRACT

In this paper we prove some common fixed point results for three mappings in vector b-metric space. Our results extend and improve some well-known results in literature. We also give an example to justify our results.

KEYWORDS : b-metric space, contraction mapping theorem, vector b-metric space, Rieszspace, weakly compatible.

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1. INTRODUCTION

Common fixed point theorems for three mappings in metric space were studied by Latpate et al[1]Similar results can be seen in Abbas et al[2],Arshad et al[3],

Jungck[4]and Rahimi et al[5].Further ,these results were extended for vector metric space by Altun and Cevik[6].We extend some of the results of fixed point for three mappings defined on vector b-metric space which is aRiesz space valued metric space. Vector b-metric space was defined by Petre[7] in 2014 by defining b-metric on vector metric space. We recall the basic concepts and definitions introduced by Altun andCevik[8] and Petre[7].

We follow notions and terminology by AliprantisandBorder [9], Luxemburg andZannen[10] for Riesz spaces.

A partially ordered set $(E, \leq)$ is a lattice if each pair of elements has a supremum and infimum.A real linear space E with an order relation $\leq$ on E which is compatible with the algebraic structure of E is called an ordered linear space.Riesz space is an ordered vector space and at the same time a lattice also. Let E be a Riesz space with the positive cone

$E_+ = \{x \in E : x \geq 0\}$. For an element $x \in E$, the absolute value $|x|$, the positive part x^+, the negative part x^- are defined as $|x| = x \vee (-x)$, $x^+ = x \vee 0$,$x^- = (-x)\vee 0$ respectively.

If every non–empty subset of E which is bounded above has a supremum, then E is called Dedekind complete or order complete. The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a\in E_+$.

Let E be a Riesz space. A sequence (b_n) is said to be order convergent or o–convergent to b if there is a sequence (a_n) in E satisfying $a_n\downarrow 0$ and $|b_n - b|\leq a_n$ for all n, written as $b_n \xrightarrow{0} b$ or $o.\lim b_n = b$.

A sequence (b_n) is said to be order Cauchy (o-Cauchy) if there exists a sequence (a_n) in E such that $a_n\downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all n and p.

A Riesz space E is said to be o-Cauchy complete if every o-Cauchy sequence is o-convergent.

DEFINITION 1.1[10] : Let X be a non–empty set and E be a Riesz space. Then function $d : X \times X \to E$ is said to be a vector metric (or E–metric) if it satisfies the following properties :

(a) $d(x, y) = o$ if and only if $x = y$

(b) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Also the triple (X, d, E) is said to be a vector metric space. Vector metric space is generalization of metric space. For arbitrary elements x, y, z, w of a vector metric space, the following statements are satisfied :

(i) $0 \leq d(x, y)$ (ii) $d(x, y) = d(y, x)$

(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$

(iv) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$

A sequence (x_n) in a vector metric space (X, d, E) vectorial converges (E-converges) to some $x \in E$, written as $x_n \xrightarrow{d.E} x$ if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n.

A sequence (x_n) is called E-cauchy sequence whenever there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p.

A vector metric space X is called E-complete if each E–cauchy sequence in X, E converges to a limit in X.

For more detailed discussion regarding vector metric spaces we refer to [6,8].

When E = R, the concepts of vectorial convergence and metric convergence, E-cauchy sequence and Cauchy sequence in metric are same.

When also X = E and d is the absolute valued vector metric on X, then the concept of vectorial convergence and convergence in order are the same.

DEFINITION 1.2: Let X be a non–empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is called a b-metric provided that, for all $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq s[d(y, x) + d(y, z)]$

A pair (X, d) is called a b-metric space. It is clear from definition that b-metric space is an extension of usual metric space.

Several authors have investigated fixed point theorems on b-metric spaces, one can see [11, 12].

Petre[7] defined E-b-metric space or vector b-metric space as follows:

DEFINITION 1.3 [7] : Let X be a nonempty set and $s \geq 1$, A functional $d : X \times X \to E_+$ is called an E-b-metric if for any $x, y, z \in X$, the following conditions are satisfied :

(a) $d(x, y) = 0$ if and only if $x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$

The triple (X, d, E) is called E-b-metric space.

EXAMPLE 1.4: Let d: $[0,1]\times[0,1]\to R^2$ defined by $d(x,y) = (\alpha|x-y|^2, \beta|x-y|^2)$ then (X,d,R^2) is E-b-metric space where $\alpha,\beta > 0$.

DEFINITION 1.5[13]: Let A and B be self maps of a set X if $y = Ax = Bx$ for some $x \in X$, then y is said to be a point of coincidence and x is said to be a coincidence point of A and B. A pair of maps A and B is called weakly compatible pair if they commute at coincidence points[8, 11].

LEMMA 1.6 [13]: If E is a Riesz space and $a \le ka$ where $a \in E_+$ and $k \in [0,1)$ then $a = 0$.

LEMMA 1.7 [14]: Let P and Q are weakly compatible self-maps on a set Y. If P and Q have a unique point of coincidence $c = Pc = Qc$, then c is the unique common fixed point of P and Q.

2. MAIN RESULTS : In this section, we prove some fixed point theorems for three mappings in vector b–metric space. Kir and Kiziltunc[12] have investigated common fixed point theorems for weakly compatible pairs for b-metric space, whereas these results on vector metric spaces have been investigated by Rad and Altun[15]

THEOREM 2.1 : Let X be E-b-metric space with E-Archimedean. Suppose the mappings $P,Q,R : X \to X$ satisfy the following conditions :

(i) for all $x, y \in X$, $d(Px, Qy) \le tM_{x,y}(P, Q, R)$ (1)

where $t < \frac{1}{s(s+1)}$ and

$M_{x,y}(P,Q,R) \in \{d(Rx, Ry), d(Px, Rx), d(Qy, Ry), d(Px, Ry), d(Qy, Rx)\}$ (2)

(ii) $P(X) \cup Q(X) \subseteq R(X)$

(iii) R(X) is an E-complete subspace of X.

Then {P,R} and {Q,R} have a unique point of coincidence in X. Moreover, if {P,R} and {Q,R} are weakly compatible, then P,Q and R have a unique fixed point in X.

PROOF : Let x_0 be arbitrary point of X. Since $P(X) \subset R(X)$ there exists $x_1 \in X$ such that $P(x_0) = Rx_1 = y_1$.

Since $Q(X) \subset R(X)$ there exists $x_2 \in X$ such that $Q(x_1) = Rx_2 = y_2$.

Continue in this manner, then there exists $x_{2n+1} \in X$ such that $P(x_{2n}) = Rx_{2n+1} = y_{2n+1}$.

there exists $x_{2n+2} \in X$ such that $Q(x_{2n+1}) = Rx_{2n+2} = y_{2n+2}$, for n = 0,1,2,3….

Firstly, show that

$$d(y_{2n+1}, y_{2n+2}) \le \beta d(y_{2n}, y_{2n+1}) \text{ for all n where } \beta < 1 \quad (3)$$

From (1), we have :

$d(y_{2n+1},y_{2n+2}) = d(Px_{2n}, Qx_{2n+1}) \le tM_{x_{2n},x_{2n+1}}(P, Q, R)$ for n= 0,1,2,3…..

Since $M_{x_{2n},x_{2n+1}}(P,Q,R) \in \{d(Rx_{2n}, Rx_{2n+1}), d(Px_{2n}, Rx_{2n}), d(Qx_{2n+1}, Rx_{2n+1}), d(Px_{2n}, Rx_{2n+1}), d(Qx_{2n+1}, Rx_{2n})\}$

$= \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}), d(y_{2n+2}, y_{2n})\}$

$= \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), \}$

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = d(y_{2n}, y_{2n+1})$, then clearly (3) holds.

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = d(y_{2n+1}, y_{2n+2})$, then according to lemma 1.6

$d(y_{2n+1}, y_{2n+2}) = 0$, and clearly (3) holds.

Finally, suppose that $M_{x_{2n},x_{2n+1}}(P,Q,R) = d(y_{2n}, y_{2n+2})$,

Then, we have

$d(y_{2n+1}, y_{2n+2}) \leq td(y_{2n},y_{2n+2}) \leq ts[d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2})]$

$(1\text{-}ts)\, d(y_{2n+1},y_{2n+2}) \leq tsd(y_{2n},y_{2n+1})$

$\leq \left(\frac{ts}{1-ts}\right) [d(y_{2n},y_{2n+1})]$

$= \beta\, d(y_{2n},y_{2n+1})$, where $\beta = \left(\frac{ts}{1-ts}\right)$

Thus $d(y_n,y_{n+1}) \leq \beta^n d(y_0, y_1)$, where $\beta \in \left\{t, \frac{ts}{1-ts}\right\}$

Therefore for all n and p,

$d(y_n,y_{n+p}) \leq s\, d(y_n,y_{n+1}) + s^2 d(y_{n+1},y_{n+2}) + s^3 d(y_{n+2},y_{n+3}) + \ldots\ldots + s^p d(y_{n+p-1},y_{n+p})$

$\leq s\, \beta^n d(y_0,y_1) + s^2 \beta^{n+1} d(y_0,y_1) + \ldots\ldots\ldots\ldots + s^p \beta^{n+p-1} d(y_0,y_1)$

$= s\beta^n \left(\frac{1-(s\beta)^p}{1-s\beta}\right) d(y_0,y_1)$

$\leq \left(\frac{s\beta^n}{1-s\beta}\right) d(y_0,y_1)$

Since E is Archimedean, then (y_n) is E-Cauchy sequence. Suppose that R(X) is E-complete, there exists a $p \in R(X)$ such that

$Rx_{2n} = y_{2n} \xrightarrow{d.E.} p$ and $Rx_{2n+1} = y_{2n+1} \xrightarrow{d.E.} p$

Hence there exists a sequence (c_n) in E such that $c_n \downarrow 0$ and $d(Rx_{2n},p) \leq c_n$,

$d(Rx_{2n+1}, p) \leq c_{n+1}$. Since $p \in R(X)$, there exists $k \in X$ such that $Rk = p$. Now we prove that $Qk = p$

For this, consider

$d(p,Qk) \leq sd(p, Px_{2n}) + sd(Px_{2n},Qk)$

$\leq sc_{n+1} + stM_{x_{2n},k}(P,Q,R)$

where $M_{x_{2n},k}(P,Q,R) \in \{d(Rx_{2n},R_k), d(Px_{2n},Rx_{2n}), d(Qk,Rk), d(Px_{2n}, Rk), d(Qk,Rx_{2n})\}$

$= \{d(y_{2n}, p), d(y_{2n+1}, y_{2n}), d(Qk, p), d(y_{2n+1},p), d(Qk, y_{2n})\}$ for all n.

There are five possibilities:

Case 1: $d(p, Qk) \le sc_{n+1} + st\, d(y_{2n},p) \le sc_{n+1} + stc_n \le s(t+1)\, c_n$.

Case 2: $d(p, Qk) \le sc_{n+1} + st\, d(y_{2n+1}, y_{2n}) \le sc_{n+1} + st\, [sd(y_{2n+1},p) + sd(p,y_{2n})]$

$\le sc_{n+1} + st[sc_{n+1} + sc_n] \le s(2st+1)\, c_n$.

Case 3: $d(p, Qk) \le sc_{n+1} + std(p,Qk)$

$(1 - st)d(p, Qk) \le sc_{n+1}$

$$d(p, Qk) \le \left(\frac{s}{1-st}\right)c_{n+1}$$

Case 4: $d(p, Qk) \le sc_{n+1} + st\, d(y_{2n+1},p)$

$\le sc_{n+1} + stc_{n+1} \le s(t+1)\, c_n$.

Case 5 : $d(p, Qk) \le sc_{n+1} + std(Qk,y_{2n})$

$\le sc_{n+1} + st[sd(Qk,p)+ sd(p,y_{2n})]$

$(1 - s^2t)\, d(p, Qk) \le sc_{n+1} + s^2td(p,y_{2n})$

$(1-s^2t)\, d(p, Qk) \le sc_{n+1} + s^2tc_n$

$$d(p, Qk) \le \left(\frac{s(1+st)}{1-s^2t}\right)c_n$$

Since the infimum of the sequences on the right hand side are zero, then $d(p,Qk) = 0$, that is $Qk = p$. Therefore $Qk = Rk = p$, i.e. p is a point of coincidence of mappings Q, R and k is a coincidence point of mappings Q and R.

Now we show that $Pk = p$, consider

$d(Pk,p) \le sd(Pk, Qx_{2n+1}) + sd(Qx_{2n+1},p) \le sc_{n+1} + stM_{x_k,2n+1}(P,Q,R)$

where $M_{x_k,2n+1}(P,Q, R) \in \{d(Rk,Rx_{2n+1}), d(Pk,Rk), d(Qx_{2n+1},Rx_{2n+1}), d(Pk, Rx_{2n+1}),$
$d(Qx_{2n+1}, Rk)\}$

$= \{d(p,y_{2n+1}), d(Pk, p), d(y_{2n+2}, y_{2n+1}), d(Pk,y_{2n+1}), d(Qx_{2n+1},p)\}$ for all n.

There are five possibilities:

Case 1: $d(Pk, p) \le sc_{n+1} + std(p,y_{2n+1}) \le sc_{n+1} + stc_{n+1} \le s(t+1)\, c_n$.

Case 2: $d(Pk,p) \le sc_{n+1} + std(Pk,p)$

$(1-st)\, d(Pk, p) \le sc_{n+1}$

$$d(Pk,p) \le \left(\frac{s}{1-st}\right)c_{n+1}$$

Case 3: $d(Pk,p) \leq sc_{n+1} + std(y_{2n+2}, y_{2n+1}) \leq sc_{n+1} + st[sd(y_{2n+2}, p) + sd(p,y_{2n+1,})]$

$d(Pk,p) \leq sc_{n+1} + st[sc_{n+2} + sc_{n+1}]$

$d(Pk,p) \leq sc_{n+1} + s^2tsc_{n+1} \leq s(st+1)\, c_{n+1}$.

Case 4: $d(Pk, p) \leq sc_{n+1} + std(Pk,y_{2n+1})$

$\leq sc_{n+1} + st[sd(Pk,p) + sd(p,y_{2n+1})] \leq sc_{n+1} + s^2td(Pk,p) + s^2tc_{n+1}$

$(1- s^2t)d(Pk, p) \leq s(1+st)\, c_{n+1}$.

$$d(Pk,p) \leq \left(\frac{s(1+st)}{(1-s^2t)}\right)c_{n+1}$$

Case 5 : $d(Pk,p) \leq sc_{n+1} + std(Qx_{2n+1}, p)$

$\leq sc_{n+1} + stc_{n+1} \leq s(1 + t)c_{n+1}$

Since the infimum of thesequences on the right hand side are zero, then $d(Pk,p) = 0$, that is $Pk = p$. Therefore $Pk = Rk = p$, i.e. p is a point of coincidence of mappings P, R and k is a coincidence point of mappings P and R.

Now it remains to prove that p is a unique point of coincidence of pairs {P,R} and {Q,R}.

Let p' be also a point of coincidence of these three mappings, then $Pk' = Qk' = Rk' = p'$,

for $k' \in X$, we have,

$d(p, p') = d(Pk, Qk') \leq tM_{k,k'}(P,Q,R)$

where $M_{k,k'}(P,Q,R) \in \{d(Rk, Rk'), d(Pk,Rk), d(Qk',Rk'), d(Pk, Rk'), d(Qk',Rk)\}$

$= \{0, d(p,p')\}$

If {P,R} and {Q,R} are weakly compatible, then p is a unique common fixed point of P,Q and R.

COROLLARY 2.2 : Let X be E-b-metric space with E Archimedean. Suppose the mappingsP,R : $X \rightarrow X$ satisfy the following conditions :

(i) for all $x, y \in X$, $d(Px, Py) \leq tM_{x,y}(P, R)$ (4)

where $t < \dfrac{1}{s(s+1)}$

$M_{x,y}(P,R) \in \{d(Rx, Ry), d(Px, Rx), d(Py, Ry), d(Px, Ry), d(Py, Rx)\}$ (5)

(ii) $P(X) \subseteq R(X)$

(iii) R(X) is E-complete subspace of X.

Then {P,R} have a unique point of coincidence in X. Moreover, if {P, R} are weakly compatible, then they have a unique fixed point in X.

EXAMPLIE 2.3 : Let $E=R^2$ with coordinatewise ordering defined by $(x_1,y_1) \leq (x_2,y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$, X = R and $d(x, y) = (|x-y|^2, c|x-y|^2)$ with $c > 0$.

Define the mappings $Px = x^2 + 3$, $Rx = 2x^2$.

For all x, y $\in$ X, we have

$d(Px, Py) = \frac{1}{2} d(Rx, Ry) \leq tM_{x,y}(P,R)$

with $M_{x,y}(P, R) = d(Rx, Ry)$ for $k \in \left[\frac{1}{2}, 1\right)$.

Moreover, $P(X) = [3, \infty) \subset [0, \infty) = R(X)$.

THEOREM 2.4 : Let X be E-b-metric space with E Archimedean. Suppose the mappings P,Q,R : X $\rightarrow$X satisfy the following conditions :

(i) for all x, y $\in$ X, $d(Px,Qy) \leq tM_{x,y}(P,Q,R)$ (6)

where $t < \frac{2}{s(s+2)}$ and

$M_{x,y}(P,Q,R) \in \{\frac{1}{2}[d(Rx, Ry) + d(Px, Rx)], \frac{1}{2}[d(Rx, Ry) + d(Px, Ry)], \frac{1}{2}[d(Rx, Ry) + d(Qy, Rx)], \frac{1}{2}[d(Rx, Ry) + d(Qy, Ry)], \frac{1}{2}[d(Px, Rx) + d(Qy, Ry)], \frac{1}{2}[d(Px, Ry) + d(Qy, Rx)]\}$ (7)

(ii) $P(X) \cup Q(X) \subseteq R(X)$

(iii) R(X) is an E-complete subspace of X.

Then {P,R} and {Q,R} have a unique common point of coincidence in X. Moreover, if {P,R} and {Q,R} are weakly compatible, then they have a unique fixed point in X.

PROOF : We define the sequence $\{x_n\}$ and $\{y_n\}$ as in proof of theorem 2.1

Firstly, show that

$$d(y_{2n+1}, y_{2n+2}) \leq \beta d(y_{2n}, y_{2n+1}) \text{ for all n.} \quad (8)$$

From (6), we have :

$d(y_{2n+1}, y_{2n+2}) = d(Px_{2n}, Qx_{2n+1}) \leq tM_{x_{2n},x_{2n+1}}(P, Q, R)$ for n= 0,1,2,3…..

Since

$M_{x_{2n},x_{2n+1}}(P, Q, R) \in \{\frac{1}{2}[d(Rx_{2n}, Rx_{2n+1}) + d(Px_{2n}, Rx_{2n})], \frac{1}{2}[d(Rx_{2n}, Rx_{2n+1}) + d(Px_{2n}, Rx_{2n+1})], \frac{1}{2}[d(Rx_{2n}, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n})], \frac{1}{2}[d(Rx_{2n}, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2}[d(Px_{2n}, Rx_{2n}) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2}[d(Px_{2n}, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n})]\}$

$= \{ \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})], \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+1})], \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})], \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2}[d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})],$

$\frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]\}$

$= \{d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n}, y_{2n+1})], \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})], \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2}[d(y_{2n}, y_{2n+2})]\}$

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = d(y_{2n}, y_{2n+1})$ or $\frac{1}{2}[d(y_{2n}, y_{2n+1})]$ then clearly (8) holds.

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]$

Then $d(y_{2n+1}, y_{2n+2}) \le \frac{t}{2}[d(y_{2n}, y_{2n+1})] + \frac{t}{2}[d(y_{2n+2}, y_{2n})]$

$$\le \frac{t}{2}[d(y_{2n}, y_{2n+1})] + \frac{t}{2}[sd(y_{2n+2}, y_{2n+1}) + sd(y_{2n+1}, y_{2n})]$$

$$\left(1-\frac{st}{2}\right)d(y_{2n+1}, y_{2n+2}) \le (1+s)\frac{t}{2}[d(y_{2n}, y_{2n+1})]$$

$$d(y_{2n+1}, y_{2n+2}) \le \frac{t}{2}\left(\frac{1+s}{1-\frac{st}{2}}\right)[d(y_{2n}, y_{2n+1})] \le \beta'[d(y_{2n}, y_{2n+1})], \quad \text{where } \beta' = \frac{t}{2}\left(\frac{1+s}{1-\frac{st}{2}}\right)$$

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = \frac{1}{2}[d(y_{2n}, y_{2n+1})+d(y_{2n+2}, y_{2n+1})]$

Then $d(y_{2n+1}, y_{2n+2}) \le \frac{t}{2}[d(y_{2n}, y_{2n+1})] + \frac{t}{2}[d(y_{2n+2}, y_{2n+1})]$

$$\left(1-\frac{t}{2}\right)d(y_{2n+1}, y_{2n+2}) \le \frac{t}{2}[d(y_{2n}, y_{2n+1})]$$

$$d(y_{2n+1}, y_{2n+2}) \le \left(\frac{\frac{t}{2}}{1-\frac{t}{2}}\right)[d(y_{2n}, y_{2n+1})] \le \beta''[d(y_{2n}, y_{2n+1})], \quad \text{where } \beta'' = \left(\frac{\frac{t}{2}}{1-\frac{t}{2}}\right)$$

If $M_{x_{2n},x_{2n+1}}(P,Q,R) = \frac{1}{2}[d(y_{2n}, y_{2n+2})]$

Then $d(y_{2n+1}, y_{2n+2}) \le \frac{t}{2}[sd(y_{2n}, y_{2n+1}) + sd(y_{2n+1}, y_{2n+2})]$

$d(y_{2n+1}, y_{2n+2}) \leq \left(\dfrac{\frac{st}{2}}{1-\frac{st}{2}}\right)$ $[d(y_{2n}, y_{2n+1})] \leq \beta''' [d(y_{2n}, y_{2n+1})]$, where $\beta''' = \left(\dfrac{\frac{st}{2}}{1-\frac{st}{2}}\right)$.

Therefore $d(y_n,y_{n+1}) \leq (\beta''')^n d(y_0,y_1)$ (9)

By using (9), for all n and p, we have

$d(y_n,y_{n+p}) \leq s\, d(y_n,y_{n+1}) + s^2 d(y_{n+1},y_{n+2}) + \ldots + s^p d(y_{n+p-1},y_{n+p})$

$\leq s\, (\beta''')^n d(y_0,y_1) + s^2 (\beta''')^{n+1} d(y_0,y_1) + \ldots + s^{n+p}(\beta''')^{n+p-1} d(y_0,y_1)$

$= s(\beta''')^n \left(\dfrac{1-(s\beta''')^p}{1-(s\beta''')}\right) d(y_0,y_1) \leq \left(\dfrac{s(\beta''')^n}{1-s\beta'''}\right) d(y_0,y_1)$

Since E is Archimedean, then (y_n) is E-Cauchy sequence. Suppose that R(X) is E-complete, there exists a $q \in R(X)$ such that

$Rx_{2n} = y_{2n} \xrightarrow{d.E.} q$ and $Rx_{2n+1} = y_{2n+1} \xrightarrow{d.E.} q$

Hence there exists a sequence (c_n) in E such that $c_n \downarrow 0$ and $d(Rx_{2n}, q) \leq c_n$,

$d(Rx_{2n+1},q) \leq c_{n+1}$. Since $q \in R(X)$, there exists $k \in X$ such that $Rk = q$. Now we prove that $Qk = q$

For this, consider

$d(q,Qk) \leq sd(q, Px_{2n}) + sd(Px_{2n},Qk) \leq sc_{n+1} + stM_{x_{2n},k}(P,Q,R)$

where $M_{x_{2n},k}(P, Q, R) \in \{ \frac{1}{2}[d(Rx_{2n}, Rk) + d(Px_{2n}, Rx_{2n})], \frac{1}{2}[d(Rx_{2n}, Rk) + d(Px_{2n}, Rk)],$

$\frac{1}{2}[d(Rx_{2n}, Rk) + d(Qk, Rx_{2n})], \frac{1}{2}[d(Rx_{2n}, Rk) + d(Qk, Rk)], \frac{1}{2}[d(Px_{2n}, Rx_{2n}) + d(Qk, Rk)], \frac{1}{2}$

$[d(Px_{2n}, Rk) + d(Qk, Rx_{2n})]\}$

$= \{ \frac{1}{2}[d(y_{2n}, q) + d(y_{2n+1}, y_{2n})], \frac{1}{2}[d(y_{2n}, q) + d(y_{2n+1}, q)], \frac{1}{2}[d(y_{2n}, q) + d(Qk, y_{2n})],$

$\frac{1}{2}[d(y_{2n}, q) + d(Qk, q)], \frac{1}{2}[d(y_{2n+1}, y_{2n}) + d(Qk, q)], \frac{1}{2}[d(y_{2n+1}, q) + d(Qk, y_{2n})]\}$

There are six possibilities:

Case 1: $d(q, Qk) \leq sc_{n+1} + \frac{st}{2}[d(y_{2n}, q)+d(y_{2n+1}, y_{2n})]$

$\leq sc_{n+1} + \frac{st}{2} c_n + \frac{st}{2}[sd(y_{2n+1},q) + sd(q,y_{2n})]$

$\leq sc_{n+1} + \frac{st}{2} c_n + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} sc_n$

$\leq s(1+\frac{t}{2} + st)c_n$

Case 2: $d(q, Qk) \le sc_{n+1} + \frac{st}{2}[d(y_{2n}, q)+d(y_{2n+1}, q)]$

$\le sc_{n+1} + \frac{st}{2} c_n + \frac{st}{2} c_{n+1} \le s(t+1)\, c_n.$

Case 3: $d(q, Qk) \le sc_{n+1} + \frac{st}{2}[d(y_{2n}, q)+ d(Qk,y_{2n})]$

$$\le sc_{n+1} + \frac{st}{2} c_n + \frac{st}{2}[sd(Qk,q) + sd(q,y_{2n})]$$

$$\left(1-\frac{s^2t}{2}\right) d(q, Qk) \le sc_{n+1} + \frac{st}{2} c_n + \frac{s^2t}{2} c_n$$

$$d(q, Qk) \le s\left(\frac{1+\frac{t}{2}+\frac{st}{2}}{1-\frac{s^2t}{2}}\right) c_n$$

Case4 : $d(q, Qk) \le sc_{n+1} + \frac{st}{2}[d(y_{2n}, q)+d(Qk,q)]$

$$\left(1-\frac{st}{2}\right) d(q, Qk) \le sc_{n+1} + \frac{st}{2} c_n$$

$$d(q, Qk) \le s\left(\frac{1+\frac{t}{2}}{1-\frac{st}{2}}\right) c_n$$

Case 5: $d(q, Qk) \le sc_{n+1} + \frac{st}{2}[d(y_{2n+1}, y_{2n})+d(Qk, q)]$

$$\left(1-\frac{st}{2}\right) d(q, Qk) \le sc_{n+1} + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} c_n$$

$$d(q, Qk) \le s\left(\frac{1+st}{1-\frac{st}{2}}\right) c_n$$

Case 6: $d(q, Qk) \le sc_{n+1} + \frac{st}{2}[d(y_{2n+1},q) + d(Qk, y_{2n})]$

$$\le sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2}[sd(Qk,q) + sd(q,y_{2n})]$$

$$\left(1-\frac{s^2t}{2}\right) d(q, Qk) \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2t}{2} c_n$$

$$d(q, Qk) \leq s \left(\frac{1+\frac{t}{2}+\frac{st}{2}}{1-\frac{s^2t}{2}}\right) c_n,$$

Since the infimum of the sequences on the right hand side are zero, therefore d(q,Qk) = 0, that is Qk = q. Therefore Qk = Rk = q i.e. q is a point of coincidence of mappings Q, R and k is a coincidence point of mappings Q and R.

Now we show that Pk = q,

Consider, $d(Pk,q) \leq sd(Pk, Qx_{2n+1}) + sd(Qx_{2n+1}, q) \leq sc_{n+1} + stM_{x_k,2n+1}(P, Q, R)$

where $M_{x_k,2n+1}(P,Q, R) \in \{\frac{1}{2}[d(Rk, Rx_{2n+1}) + d(Pk, Rk)], \frac{1}{2}[d(Rk, Rx_{2n+1}) + d(Pk, Rx_{2n+1})], \frac{1}{2}$

$[d(Rk, Rx_{2n+1}) + d(Qx_{2n+1}, Rk)], \frac{1}{2}[d(Rk, Rx_{2n+1}) + d(Qx_{2n+1}, Rx_{2n+1})],$

$\frac{1}{2}[d(Pk, Rk) + d(Qx_{2n+1}, Rx_{2n+1})], \frac{1}{2}[d(Pk, Rx_{2n+1}) + d(Qx_{2n+1}, Rk)]\}$

$= \{\frac{1}{2}[d(q, y_{2n+1}) + d(Pk, q)], \frac{1}{2}[d(q, y_{2n+1}) + d(Pk, y_{2n+1})], \frac{1}{2}[d(q, y_{2n+1}) + d(y_{2n+2}, q)],$

$\frac{1}{2}[d(q, y_{2n+2}) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2}[d(Pk, q) + d(y_{2n+2}, y_{2n+1})], \frac{1}{2}[d(Pk, y_{2n+1}) + d(y_{2n+2}, q)]\}$

There are six possibilities:

Case 1: $d(Pk, q) \leq sc_{n+1} + \frac{st}{2}[d(q, y_{2n+1}) + d(Pk, q)]$

$$\left(1-\frac{st}{2}\right) d(Pk, q) \leq sc_{n+1} + \frac{st}{2} c_{n+1}$$

$$d(Pk, q) \leq s\left(\frac{1+\frac{t}{2}}{\left(1-\frac{st}{2}\right)}\right) c_{n+1}$$

Case 2: $d(Pk, q) \leq sc_{n+1} + \frac{st}{2}[d(q, y_{2n+1}) + d(Pk, y_{2n+1})]$

$$d(Pk, q) \leq sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2}[sd(Pk, q) + sd(q, y_{2n+1})]$$

$$\left(1-\frac{s^2t}{2}\right) d(Pk, q) \le sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}$$

$$d(Pk, q) \le s\left(\frac{1+\frac{t}{2}+\frac{st}{2}}{1-\frac{s^2t}{2}}\right) c_n$$

Case 3: $d(Pk, q) \le sc_{n+1} + \frac{st}{2}[d(q, y_{2n+1}) + d(y_{2n+2}, q)] \le sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2} c_{n+1}$

$$d(Pk, q) \le s(1+t)c_{n+1}$$

Case 4: $d(Pk, q) \le sc_{n+1} + \frac{st}{2}[d(q, y_{2n+1}) + d(y_{2n+2}, y_{2n+1})]$

$$\le sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{st}{2}[sd(y_{2n+2}, q) + sd(y_{2n+1}, q)]$$

$$\le sc_{n+1} + \frac{st}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}$$

$$\le s(1 + st + \frac{t}{2})c_{n+1}$$

Case 5 : $d(Pk, q) \le sc_{n+1} + \frac{st}{2}[d(Pk, q) + d(y_{2n+2}, y_{2n+1})$

$$\le sc_{n+1} + \frac{st}{2}[(Pk,q)] + \frac{st}{2}[sd(y_{2n+2}, q) + sd(q, y_{2n+1})]$$

$$\left(1-\frac{st}{2}\right) d(Pk, q) \le sc_{n+1} + \frac{s^2t}{2} c_{n+1} + \frac{s^2t}{2} c_{n+1}$$

$$d(Pk, q) \le s\left(\frac{1+st}{1-\frac{st}{2}}\right) c_{n+1}$$

Case 6: $d(Pk, q) \le sc_{n+1} + \frac{st}{2}[d(Pk, y_{2n+1}) + d(y_{2n+2}, q)]$

$$d(Pk, q) \le sc_{n+1} + \frac{st}{2}[sd(Pk, q) + sd(q, y_{2n+1})] + \frac{st}{2} c_{n+1}$$

$$\left(1-\frac{s^2t}{2}\right) d(Pk, q) \le s\left(\frac{1+\frac{t}{2}+\frac{st}{2}}{1-\frac{s^2t}{2}}\right) c_{n+1}$$

Since the infimum of the sequences on the right hand side are zero, therefore d(Pk, q) = 0, that is Pk = q. Therefore Pk = Rk = q, i.e. n is a point of coincidence of mappings P and R. Thus k is a coincidence point of mappings P and R.

Now it remains to prove that q is a unique point of coincidence of pairs {P, R} and {Q, R}.

Let q' be also a point of coincidence of these three mappings, then $Pk' = Qk' = Tk' = q'$,

for $k' \in X$, we have,

$d(q, q') = d(Pk, Qk') \leq tM_{k,k'}(P, Q, R)$

where $M_{k,k'}(P, Q, R) \in \{ \frac{1}{2}[d(Rk, Rk') + d(Pk, Rk)], \frac{1}{2}[d(Rk, Rk') + d(Pk, Rk')],$

$\frac{1}{2}[d(Rk, Rk') + d(Qk', Rk)], \frac{1}{2}[d(Rk, Rk') + d(Qk', Rk')], \frac{1}{2}[d(Pk, Rk) + d(Qk', Rk')],$

$\frac{1}{2}[d(Pk, Rk') + d(Qk', Rk)]\}$

$= \{0, d(q, q')\}$

Hence d(q, q') = 0 i.e. q = q'

If {P, R} and {Q, R} are weakly compatible, then q is a unique common fixed point of P, Q and R.

3.RESULTS AND DISCUSSION

In 2016, Rad and Altun[15]proved some common fixed point results for three mappings on vector metric spaces. They proved the following results:

THEOREM 3.1 : Let X be a vector metric space with E-Archimedean. Suppose the mappings $f,g,T : X \to X$ satisfy the following conditions :

(i) for all $x, y \in X$, $d(fx, gy) \leq ku_{x,y}(f, g, T)$ (10)

where $k \in (0, 1)$ is a constant and

$u_{x,y}(f,g,T) \in \{d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Tx)]$(11)

(ii) $f(X) \cup g(X) \subseteq T(X)$

(iii) one of f(X), g(X) or T(X) isaE-complete subspace of X.

Then {f,T} and {g,T} have a unique point of coincidence in X. Moreover, if {f,T} and {g,T} are weakly compatible, then f,g and T have a unique common fixed point in X

where $k \in (0, 1]$. (12)

$u_{x,y}(f, g) \in \{d(fx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$ (13)

(ii) $f(X) \subseteq T(X)$

(iii) one of f(X) or T(X) isaE-complete subspace of X.

Then {f, T} have a unique point of coincidence in X. Moreover, if {f, T} are weakly compatible, then f and T have a unique common fixed point in X.

In 2017, Latpate[1] proved the results for three mappings on complete metric spaces. He proved the following result:

Let (X, d) be a complete Metric space and Let A be a nonempty closed subset of X.

Let P, Q: A→A be such that

$$d(P_x, Q_y) \leq \frac{1}{2}[d(R_x, Q_y) + d(R_y, P_x) + d(S_x, R_y)] - \psi[d(R_x, Q_y) + d(R_y, P_x)] \quad (14)$$

For any (x, y)∈X × X, where a function ψ: $[0, \infty)^2 \rightarrow [0, \infty)$ is continuous and ψ(x, y) = 0 iff x = y = 0 and R: A → Xwhich satisfies the following condition.

(i) PA ⊆ RA and QA ⊆ RA

(ii) The pair of mappings (P,R) and (Q, R) are weakly compatible.

(iii) R(A) is closed subset of X.

Then P,R and Q have unique common fixed point.

Motivated by their results, we have proved similar results for three mappings on E-b-metric spaces.

Further, these results can be investigated for four and six mappings on E-b-metric space.

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