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Investigation of Inter Relation Among Different Types of Continuous Functions on Convex Topological Space

Bijoy Samanta

Assistant Professor in Mathematics Department of Mathematics , Shibpur Dinobundhoo Institution
(College)412/1 , G.T. Road (S) , Shibpur , Howrah – 711 102 , W.B. , India
E-mail : bijoysamanta@yahoo.com

ABSTRACT

In this paper some continuous functions has been introduced using both the topology τ and convexity \mathcal{C} on the same underlying set X where (X, τ, \mathcal{C}) is termed as convex topological space and inter relation among them are also investigated .

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***Corresponding author:**

Bijoy Samanta

Assistant Professor in Mathematics

Department of Mathematics , Shibpur Dinobundhoo Institution

(College)412/1 , G.T. Road (S) ,

Shibpur , Howrah – 711102 , W.B. , India

E-mail : bijoysamanta@yahoo.com

1. INTRODUCTION

The development of “abstract convexity” has emanated from different sources in different ways ; the first type of development basically banked on generalization of particular problems such as separation of convex sets¹ , *extremality*^{2,3} or continuous *selection*⁴ . The second type of development lay before the reader such axiomatizations , which in every case of design , express particular point of view of convexity . With the view point of generalized topology which enters into convexity via the closure or hull operator , Schmidt and Hammer, , introduced some axioms to explain abstract convexity . The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birchoff and Frink , Schmidt, Hammer .

The axiomatizations as proposed by M.L.J. Van De Vel in his paper *Theory of Convex Structure*⁵ will be followed Through out in this paper .

The author has discussed in “ Topology and Convexity on the same set⁶ ” and introduced the compatibility of the topology with a convexity on the same underlying set . At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet (X, τ, \mathcal{C}) and call it convex topological space only to bring back “compatibility” in another way subsequently . With this compatibility , Van De Vel has called the triplet (X, τ, \mathcal{C}) a topological convex structure .

In this paper , Art. 2 deals with some early definitions , results and in Art. 3 we have discussed mainly inter relation among different types of continuous functions .

2. PREREQUISITES :

Definition 2.1⁶ : Let X be a non empty set . A family \mathcal{C} of subsets of the set X is called a convexity on X if

1. $\phi, X \in \mathcal{C}$
2. \mathcal{C} is stable for intersection , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty , then $\cap \mathcal{D} \in \mathcal{C}$
3. \mathcal{C} is stable for nested unions , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty and totally ordered by set inclusion , then $\cup \mathcal{D} \in \mathcal{C}$.

The pair (X, \mathcal{C}) is called a convex structure . The members of \mathcal{C} are called convex sets and their complements are called concave sets .

Definition 2.2⁶ : Let \mathcal{C} be a convexity on set X . Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by $co(A) = \cap \{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3⁶ : Let (X, \mathcal{C}) be a convex structure and let Y be a subset of X . The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y ; called the relative convexity of Y .

Note 2.4⁶ : The hull operator co_Y of a subspace (Y, \mathcal{C}_Y) satisfy the following :

$$\forall A \subseteq Y : co_Y(A) = co(A) \cap Y .$$

Definition 2.5⁶: Let (X, \mathcal{C}) be a convex structure and let τ be a topology on X . Then τ is said to be compatible with the convex structure (X, \mathcal{C}) if all polytopes of \mathcal{C} are closed in τ where polytopes means convex hull of a finite set. Also the triplet (X, τ, \mathcal{C}) is then called topological convex structure.

Note 2.6⁶: Let (X, τ, \mathcal{C}) be a topological convex structure. Then collection of all closed sets in (X, τ) are subset of \mathcal{C} .

Definition 2.7⁷: Let (X, τ) be a topological space and let \mathcal{C} be a convexity on X . Then the triplet (X, τ, \mathcal{C}) is called a convex topological space (CTS in short).

Definition 2.8⁸: Let (X, τ, \mathcal{C}) be a convex topological space. A set $P \subseteq X$ is said to be C-regular open if $P = int(co(P))$.

Result 2.9⁸: Let A be a subset of a convex topological space (X, τ, \mathcal{C}) . Then $int(co(A))$ is a C-regular open set.

Note 2.10⁸: In a convex topological space (X, τ, \mathcal{C}) for any subset A of X , the set $int(co(A))$ is a C-regular open set. Also a subset B of X is called C-regular closed set if its complement is C-regular open set.

Definition 2.11⁸: [8] Let (X, τ, \mathcal{C}) be a convex topological space. Let S be a subset of X and $x \in X$.

- (a) x is called δ_* - \mathcal{C} cluster point of S if $S \cap int(co(U)) \neq \emptyset$, for each open nbd. U of x .
- (b) The family of all δ_* - \mathcal{C} cluster points of S is called the δ_* - \mathcal{C} closure of S and is denoted by $[S]_{\delta_*}$.
- (c) A subset S of X is called δ_* - \mathcal{C} closed if $[S]_{\delta_*} = S$.

The complement of a δ_* - \mathcal{C} closed set is said to be a δ_* - \mathcal{C} open set.

Definition 2.12⁸: Let (X, τ, \mathcal{C}_1) and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is said to be δ_* - \mathcal{C} continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f(int(co(U))) \subseteq int(co(V))$.

Result 2.13⁸: A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is δ_* - \mathcal{C} continuous iff for each $x \in X$ and each C-regular open set V containing $f(x)$, there exists a C-regular open set U containing x such that $f(U) \subseteq V$.

3. COMPARISON OF DIFFERENT TYPES OF CONTINUOUS FUNCTIONS :

Definition 3.1 : Let (X, τ, \mathcal{C}_1) and $(Y, \sigma, \mathcal{C}_2)$ be two convex topological spaces . A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is said to be strongly θ_* - \mathcal{C} continuous , $\theta_* - \mathcal{C}$ continuous , regular \mathcal{C} - continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f(\text{co}(U)) \subseteq V$, $f(\text{co}(U)) \subseteq \text{co}(V)$, $f(U) \subseteq \text{int}(\text{co}(V))$ respectively .

Remark 3.2 : For any convex topological space we have $P \subseteq \text{co}(P)$. This shows that strongly $\theta_* - \mathcal{C}$ continuous $\Rightarrow \theta_* - \mathcal{C}$ continuous .

The following example shows that the converse of the above implication may not be true in general

Example 3.3 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X\}$ and f is the identity mapping I_X on X .

Here f is $\theta_* - \mathcal{C}$ continuous on X . Also for $a \in X$ if we consider the open nbd. $\{a\}$ of $f(a)$, then there is no $U \in \tau$ such that $f(\text{co}(U)) \subseteq \{a\}$. So f is not strongly $\theta_* - \mathcal{C}$ continuous on X .

Note 3.4 : We now show that $\theta_* - \mathcal{C}$ continuous and regular \mathcal{C} - continuous are two independent concepts which follows from the next two examples .

Example 3.5 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\mathcal{C}_1 = \{\emptyset, X, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a, b\}\}$ and f is the identity mapping I_X on X .

Here f is $\theta_* - \mathcal{C}$ continuous but not regular \mathcal{C} - continuous on .

Example 3.6 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a\}\}$ and f is the identity mapping I_X on X .

Here f is regular \mathcal{C} - continuous but not $\theta_* - \mathcal{C}$ continuous on .

Theorem 3.7 : Let a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be $\theta_* - \mathcal{C}$ continuous and open . Then f is regular \mathcal{C} - continuous function .

Proof: Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is $\theta_* - \mathcal{C}$ continuous , there exists an open nbd. U of x such that $f(\text{co}(U)) \subseteq \text{co}(V)$. Thus $f(U) \subseteq f(\text{co}(U)) \subseteq \text{co}(V)$. Now U is an open set and f is open mapping . Thus $f(U)$ is an open set in Y which is contained in $\text{co}(V)$. So $f(U) \subseteq \text{int}(\text{co}(V))$. Consequently f is regular \mathcal{C} - continuous function .

Theorem 3.8 : (a) If a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is strongly θ_* - \mathcal{C} continuous and $g : (Y, \sigma, \mathcal{C}_2) \rightarrow (Z, \gamma, \mathcal{C}_3)$ is regular \mathcal{C} -continuous, then $g \circ f : (X, \tau, \mathcal{C}_1) \rightarrow (Z, \gamma, \mathcal{C}_3)$ is δ_* - \mathcal{C} continuous.

(b) The following implications hold :

strongly θ_* - \mathcal{C} continuous $\Rightarrow \delta_*$ - \mathcal{C} continuous \Rightarrow regular \mathcal{C} -continuous .

Proof : (a) Let $x \in X$ and W be any open set containing $(g \circ f)(x)$. Since g is regular \mathcal{C} -continuous, there exists an open nbd. V of $f(x)$ in Y such that $g(V) \subseteq \text{int}(co(W))$. Again since f is strongly θ_* - \mathcal{C} continuous, there exists an open nbd. U of x in X such that $f(co(U)) \subseteq V$. Now $f(\text{int}(co(U))) \subseteq f(co(U)) \subseteq V \Rightarrow g(f(\text{int}(co(U)))) \subseteq g(f(co(U))) \subseteq g(V) \subseteq \text{int}(co(W)) \Rightarrow (g \circ f)(\text{int}(co(U))) \subseteq \text{int}(co(W))$. This shows that $g \circ f$ is δ_* - \mathcal{C} continuous.

(b) Let f be strongly θ_* - \mathcal{C} continuous. Also let $x \in X$ and V be any open nbd. of $f(x)$. Then there exists an open nbd. U of x in X such that $f(co(U)) \subseteq V \Rightarrow f(\text{int}(co(U))) \subseteq f(co(U)) \subseteq V = \text{int}(V) \subseteq \text{int}(co(V))$. Hence f is δ_* - \mathcal{C} continuous.

Again let f be δ_* - \mathcal{C} continuous. Also let $x \in X$ and V be any open nbd. of $f(x)$ in Y . Then there exists an open nbd. U of x in X such that $f(\text{int}(co(U))) \subseteq \text{int}(co(V))$. Let $W = \text{int}(co(U))$. Then W is open nbd. of x such that $f(W) \subseteq \text{int}(co(V))$. Thus f is regular \mathcal{C} -continuous.

Note 3.9 : The following examples show that none of these implications in the above theorem is reversible.

Example 3.10 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{b\}\}$, $\mathcal{C}_2 = \{\emptyset, X\}$ and f is the identity mapping I_X on X .

Here f is δ_* - \mathcal{C} continuous but not strongly θ_* - \mathcal{C} continuous on X .

Example 3.11 : Let us consider the function $f : (X, \tau, \mathcal{C}_1) \rightarrow (X, \sigma, \mathcal{C}_2)$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, $\mathcal{C}_1 = \{\emptyset, X\}$, $\sigma = \{\emptyset, X, \{a\}\}$, $\mathcal{C}_2 = \{\emptyset, X, \{a\}\}$ and f is the identity mapping I_X on X .

Here f is regular \mathcal{C} -continuous but not δ_* - \mathcal{C} continuous on X .

Definition 3.12 : A convex topological space (X, τ, \mathcal{C}) is said to be an semi \mathcal{C} -regular space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq \text{int}(co(U)) \subseteq V$.

Theorem 3.13 : For a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ the following properties are true :

- (a) If Y is an semi \mathcal{C} - regular space and f is δ_* - \mathcal{C} continuous , then f is continuous .
- (b) If X is an semi \mathcal{C} - regular space and f is regular \mathcal{C} - continuous , then f is δ_* - \mathcal{C} continuous .

Proof : (a) Let Y be an semi \mathcal{C} - regular space and $x \in X$. Then for each open nbd. V of $f(x)$, there exists an open nbd. W of $f(x)$ such that $f(x) \in W \subseteq \text{int}(\text{co}(W)) \subseteq V$. Since f is δ_* - \mathcal{C} continuous , there exists an open nbd. U of x such that $f(\text{int}(\text{co}(U))) \subseteq \text{int}(\text{co}(W))$. Since U is an open set , $f(U) = f(\text{int}(U)) \subseteq f(\text{int}(\text{co}(U))) \subseteq \text{int}(\text{co}(W)) \subseteq V$ i.e., $f(U) \subseteq V$. Hence f is continuous .

(b) Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is regular \mathcal{C} - continuous , there exists an open nbd. U of x such that $f(U) \subseteq \text{int}(\text{co}(V))$. Again since X is an semi \mathcal{C} - regular space there exists an open nbd. W of x such that $\text{int}(\text{co}(W)) \subseteq U$. Thus $f(\text{int}(\text{co}(W))) \subseteq f(U) \subseteq \text{int}(\text{co}(V))$. Hence f is δ_* - \mathcal{C} continuous .

Corollary 3.14 : If (X, τ, \mathcal{C}_1) and $(Y, \sigma, \mathcal{C}_2)$ are semi \mathcal{C} - regular spaces , then the concepts on a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$, δ_* - \mathcal{C} continuity , continuity , regular \mathcal{C} - continuity are equivalent .

Definition 3.15 : A CTS (X, τ, \mathcal{C}) is said to be an almost \mathcal{C} - regular if for each \mathcal{C} - regular closed set F and each $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 3.16 : Let (X, τ, \mathcal{C}) be an almost \mathcal{C} - regular space where τ is compatible with \mathcal{C} . Then for each $x \in X$ and each \mathcal{C} - regular open nbd. V of x , there exists a \mathcal{C} - regular open nbd. W of x such that $x \in W \subseteq \text{co}(W) \subseteq V$.

Proof : Let $x \in X$ and V be a \mathcal{C} - regular open set containing x . Then $x \notin X \setminus V$ and $X \setminus V$ is a \mathcal{C} - regular closed set . Thus there exist disjoint open sets U_1, U_2 such that $x \in U_1$ and $X \setminus V \subseteq U_2$. Now $U_1 \cap U_2 = \emptyset \Rightarrow \text{cl}(U_1) \cap U_2 = \emptyset \Rightarrow \text{co}(U_1) \cap U_2 = \emptyset$ [Since τ is compatible with \mathcal{C}] $\Rightarrow \text{co}(U_1) \subseteq X \setminus U_2 \subseteq V \Rightarrow \text{int}(\text{co}(U_1)) \subseteq V$. Let $W = \text{int}(\text{co}(U_1))$. Then $x \in W$ and W is a \mathcal{C} - regular open set . Also $W = \text{int}(\text{co}(U_1)) \subseteq \text{co}(U_1) \Rightarrow \text{co}(W) \subseteq \text{co}(U_1) \subseteq V \Rightarrow x \in W \subseteq \text{co}(W) \subseteq V$.

Theorem 3.17 : For a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ the following hold :

- (1) If Y is almost \mathcal{C} - regular space where σ is compatible with \mathcal{C}_2 and f is θ_* - \mathcal{C} continuous , then f is δ_* - \mathcal{C} continuous .
- (2) If X is almost \mathcal{C} - regular space where τ is compatible with \mathcal{C}_1 , Y is semi \mathcal{C} - regular space and f is δ_* - \mathcal{C} continuous , then f is strongly θ_* - \mathcal{C} continuous .

Proof: 1) Let $x \in X$ and V be a \mathcal{C} -regular open nbd. of $f(x)$. Then Y being almost \mathcal{C} -regular space, there exist a \mathcal{C} -regular open nbd. U of $f(x)$ such that $f(x) \in U \subseteq co(U) \subseteq V$. Since f is θ_* - \mathcal{C} continuous, there exists an open nbd. W of x such that $f(co(W)) \subseteq co(U)$. Thus $(int(co(W))) \subseteq f(co(W)) \subseteq co(U) \subseteq V$. Hence f is δ_* - \mathcal{C} continuous.

2) Let $x \in X$ and V be an open nbd. of $f(x)$. Since Y is semi \mathcal{C} -regular space, there exists an open nbd. U of $f(x)$ such that $f(x) \in U \subseteq int(co(U)) \subseteq V$. Again by the δ_* - \mathcal{C} continuity of f , there exists an open nbd. W of x such that $f(int(co(W))) \subseteq int(co(U))$. Now $int(co(W))$ is a \mathcal{C} -regular open set in X which is almost \mathcal{C} -regular. So there is \mathcal{C} -regular open nbd. P of x such that $x \in P \subseteq co(P) \subseteq int(co(W))$. This implies that $f(co(P)) \subseteq f(int(co(W))) \subseteq int(co(U)) \subseteq V$. Thus f is strongly θ_* - \mathcal{C} continuous.

Definition 3.18 : A function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ is called \mathcal{C} -regular open if for each \mathcal{C} -regular open set U of X , $f(U)$ is open in Y .

Theorem 3.19 : Let a function $f : (X, \tau, \mathcal{C}_1) \rightarrow (Y, \sigma, \mathcal{C}_2)$ be θ_* - \mathcal{C} continuous and \mathcal{C} -regular open. Then f is δ_* - \mathcal{C} continuous function.

Proof: Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is θ_* - \mathcal{C} continuous, there exists an open nbd. U of x such that $f(co(U)) \subseteq co(V)$. Thus $f(int(co(U))) \subseteq f(co(U)) \subseteq co(V)$. Now $(int(co(U)))$ is a \mathcal{C} -regular open set and f is \mathcal{C} -regular open mapping. Thus $f(int(co(U)))$ is an open set in Y which is contained in $co(V)$. So $f(int(co(U))) \subseteq int(co(V))$. Consequently f is δ_* - \mathcal{C} continuous function.

REFERENCES :

1. Ellis J. W. ; A general set separation theorem , Duke Math. J. , 1952;19: 417 – 421 .
2. Fan K. ; On the Krein – Milman Theorem , Proc. Symp. Pure Math., 1963; 7: 211 - 221 .
3. Davies E. B. ; A generalized theory of convexity , Proc. London Math. Soc., 1967; 17:644 - 652 .
4. Michael E. ; Convex structure and continuous selection , Canadian J. Math., 1959;11:556 – 575 .
5. Van De Vel M.; A selection theorem for topological convex structures , Trans. Amer. Math. Soc., 1993;336(2): 463 – 496 .
6. Van De Vel M. L. J.; Theory of convex structures , North Holland , 1993.
7. Ganguly S. and Samanta Bijoy; Some functions related to convex topological spaces , Rev. Acad. Canar. Cienc., 2006;18:9 – 21 .

8. Samanta Bijoy ; Convex Topological Space and Delta Star-C-continuous function, Communicated .
9. Ganguly S. and Samanta Bijoy; $\theta - \mathcal{C}$ somewhat continuous function , Rev. Acad. Canar. Cienc. , 2008; 20: 63– 73
10. Samanta Bijoy ; Some open sets in Convex topological Space , International Journal of Research and Analytical Reviews , 2018 ; 5(3): 493z – 495z .
11. Samanta Bijoy; A continuous function on Convex topological Space , International Journal of Research and Analytical Reviews , 2018 ; 5 (3):44a – 47a .
12. Samanta Bijoy ; Convex topological Space and some continuous functions , International Journal of Research and Analytical Reviews , 2018; 5(4):76 – 79 .
13. Birkhoff G. and Frink O. ; Represents of lattices by sets , Trans. Amer. Math. Soc., 1948; 64: 229 - 315 .
14. Hammer P. C.; General topology , symmetry and convexity , Trans . Wisc. Acad. Sci. Arts Lett. 1955;44:221 – 255 .
15. Hammer P. C. ; Extented topology ; domain finiteness , Indag. Math. , 1963; 25:200 – 212
16. Hammer P. C. ; Semi spaces and the topology from convexity , Proc. Symp. Pure Math., 1963;7:305 – 316 .
17. Schmidt J. ; Einige grundlegendle Begriffe und Satze aus der Theorie der Hullenoperatoren , Bericht Uber die Mathematiker – Tagung in Berlin , 1953; 21 – 48.