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The Upper Total Edge Domination Number of a Graph

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ABSTRACT

Let $G = (V, E)$ be a connected graph of order n. The total edge dominating set S in a connected graph G is called a *minimal total edge dominating set* if no proper subset of S is a total edge dominating set of G. The *upper total edge domination number* $\gamma_{te}^{+}(G)$ of G is the maximum cardinality of a minimal total edge dominating sets of G. Some of its general properties satisfied by this concepts are studied. It is shown that for any integer $a \geq 1$, there exists a connected graph G such that $\gamma_{te}(G) = a + 1$ and $\gamma_{te}^{+}(G) = 2a$.

KEYWORDS: domination number, total domination number, edge domination number, total edge domination number, upper total domination number.

AMS SUBJECT CLASSIFICATION: 05C69.

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1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to Chartrand [1]. $N(v) = \{u \in V(G) : uv \in E(G)\}$is called the *neighborhood* of the vertex v in G. A vertex $v$is an *extreme* vertex of a graph G if $<N(v)>$ is complete. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$and $d(v) > 1$, then we call e a *pendent edge*, u a leaf and v a *support vertex*. Let $L(G)$ be the set of all leaves of a graph G.For any connected graph G, a vertex $v \in V(G)$ is called a *cut vertex* of Gif $V - v$ is no longer connected.A set of vertices D in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G[1].A *total dominating set* of a connected graph$G$is a set$S$ of vertices of$G$ such that every vertex is adjacent to a vertex in$S$. Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number*$\gamma_t(G)$of$G$ is the minimum cardinality of total dominating sets S in G[2,3,4]. A set of edges $M$of $G$is called an *edgedominating set* if every edge of $E - M$ is adjacent to an element of M. An *edge domination number*, $\gamma_e(G)$of$G$ is the minimum cardinality of an edge dominating sets of G[5,6,7,8,9,10]. An edge dominating set S of $G$is called a *total edge dominating set* of G if $\langle S \rangle$has no isolated edges. The *total edge domination number*$\gamma_{te}(G)$of G is the minimum cardinality taken over all total edge dominating sets of G[6,11].

2. THE UPPER TOTAL EDGE DOMINATION NUMBER OF A GRAPH

Definition 2. 1.

The total edge dominating set S in a connected graph G is called a *minimal total edge dominating set* if no proper subset of S is a total edge dominating set of G. The *upper total edge domination number* $\gamma_{te}^{+}(G)$ of G is the maximum cardinality of a minimal total edge dominating sets of G.

Example 2.2

For the graph G given in Figure 1, $S_1 = \{v_1v_2, v_2v_5, v_5v_6\}$ and $S_2 = \{v_1v_7, v_1v_2, v_2v_5\}$ are the minimum total edge dominating sets of G so that $\gamma_{te}(G) = 3$. The set $S = \{v_1v_7, v_6v_7, v_2v_3, v_2v_5\}$ is a total edge dominating set of G and it is clear that no proper subset of S is the total edge dominating set of G and so S is the minimal total edge dominating set of G. Also it is easily verified that no five element or six element subset is a minimal total edge dominating set of G, it follows that $\gamma_{te}^{+}(G) = 4$.

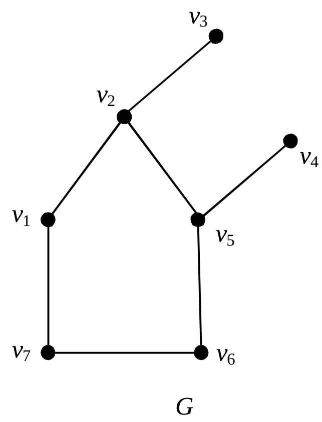


Figure 1

A graph with $\gamma_{te}^{+}(G) = 4$

Remark 2.4

Every minimum total edge dominating set of G is a minimal total edge dominating set of G and the converse is not true. For the graph G given in Figure 2.1, $S = \{v_1v_7, v_6v_7, v_2v_3, v_2v_5\}$ is a minimal total edge dominating set but not a minimum total edge dominating set of G.

Theorem 2.5

For a connected graph G, $2 \leq \gamma_{te}(G) \leq \gamma_{te}^{+}(G) \leq m$.

Proof.

We know that any total edge dominating set needs at least two edges and so $\gamma_{te}(G) \geq 2$. Since every minimal total edge dominating set is also the total edge dominating set, $\gamma_{te}(G) \leq \gamma_{te}^{+}(G)$. Also, since $E(G)$ is the total dominating set of G, it is clear that $\gamma_{te}^{+}(G) \leq m$. Thus $2 \leq \gamma_{te}(G) \leq \gamma_{te}^{+}(G) \leq m$. ■

Remark 2.6.

The bounds in Theorem 2.5 are sharp. For any graph $G = P_2$, $m = 2$, $\gamma_{te}(G) = 2$ and $\gamma_{te}^{+}(G) = 2$. Therefore $2 = \gamma_{te}(G) = \gamma_{te}^{+}(G) = m$. Also, all the inequalities in Theorem 2.5 are strict. For the graph G given in Figure 1, $\gamma_{te}(G) = 3$, $\gamma_{te}^{+}(G) = 4$ and $m = 7$ so that $2 < \gamma_{te}(G) < \gamma_{te}^{+}(G) < m$.

Theorem 2.7.

For a connected graph G, $\gamma_{te}(G) = m$ if and only if $\gamma_{te}^{+}(G) = m$.

Proof.

Let $\gamma_{te}^{+}(G) = m$. Then $S = E(G)$ is the unique minimal total edge dominating set of G. Since no proper subset of S is the total edge dominating set, it is clear that S is the unique minimum total edge dominating set of G and so $\gamma_{te}(G) = m$. The converse follows from Theorem 2.3. ■

Theorem 2.8

For complete graph $G = K_n$ $(n \geq 3)$, $\gamma_{te}^{+}(G) = 2$.

Proof.

Let S be any set of two adjacent edges of K_n. Since each edge of K_n is incident with an edge of S, it follows that S is a total edge dominating set of G so that $\gamma_{te}(G) = 2$. We show that $\gamma_{te}^{+}(G) = 2$. Suppose that $\gamma_{te}^{+}(G) \geq 3$. Then there exists a total edge dominating set S_1 such that $|S_1| \geq 3$. It is clear that S_1 contains two adjacent edges say e_1, e_2. Then $S_1{}' = \{e_1, e_2\}$ is a total edge dominating set of G, which is a contradiction. Thus $\gamma_{te}^{+}(G) = 2$. ■

Theorem 2. 9

For complete bipartite graph $G = K_{m,n}$ $(m, n \geq 2)$, $\gamma_{te}^{+}(G) = 2$.

Proof.

Let S be any set of two adjacent edges of $K_{m,n}$. Since each edge of $K_{m,n}$ is incident with an edge of S, it follows that S is a total edge dominating set of G so that $\gamma_{te}(G) = 2$. We show that $\gamma_{te}^{+}(G) = 2$. Suppose $\gamma_{te}^{+}(G) \geq 3$. Then there exists a total edge dominating set S_1 such that $|S_1| \geq 3$. It is clear that S_1 contains two adjacent edges say e_1, e_2. Then $S_1{}' = \{e_1, e_2\}$ is a total edge dominating set of G, which is a contradiction. Thus $\gamma_{te}^{+}(G) = 2$. ■

Theorem 2.10

For any graph $G = K_{1,n}$ $(n \geq 2)$, $\gamma_{te}^{+}(G) = 2$.

Proof.

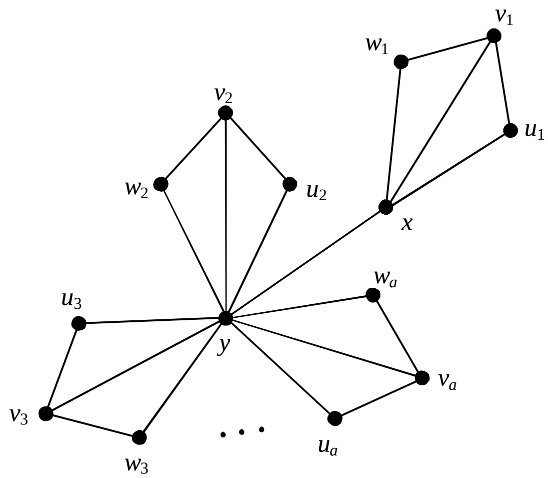
The proof is similar to Theorem 2.9. ■

Theorem 2.11

For any integer $a \geq 1$, there exists a connected graph G such that$\gamma_{te}(G) = a + 1$ and $\gamma_{te}^{+}(G) = 2a$.

Proof.

Let$P_i : u_i, v_i, w_i$ $(1 \leq i \leq a)$ be a path of order 3 and $P: x, y$ be a path of order 2. Let G be a graph obtained fromP_i $(1 \leq i \leq a)$ and P by joining y with eachu_i $(2 \leq i \leq a)$, $v_i (2 \leq i \leq a)$ and w_i $(2 \leq i \leq a)$ and also join x with u_1, v_1 and w_1. The graph G is shown in Figure 2.



G

Figure 2

First we claim that $\gamma_{te}(G) = a + 1$. It is easily observed that an edge xy belongs to every minimum total edge dominating set of G and so $\gamma_{te}(G) \geq 1$. Also it is easily seen that every minimum total edge dominating set of G contains at least one edge of each block of $G - \{x\}$ and each block of $G - \{y\}$ and so $\gamma_{te}(G) \geq a + 1$. Now $X = \{xy, xv_1, yv_2, yv_3, \dots, yv_a\}$ is a total edge dominating set of G so that $\gamma_{te}(G) = a + 1$.

Next we show that $\gamma_{te}^{+}(G) = 2a$. Now $D = \{xu_1, yu_2, yu_3, \dots, xu_a, xw_1, yw_2, yw_3, \dots, yw_a\}$ is a total edge dominating set of G. We show that D is a minimal total edge dominating set of G. Let D' be any proper subset of D. Then there exists at least one edge say $e \in D$ such that $e \notin D'$. Suppose that $e = xu_i$ for some i $(1 \leq i \leq a)$, then the edge xw_i $(1 \leq i \leq a)$ will be isolated in $\langle D' \rangle$. Therefore D' is not a total edge dominating set of G. Now, assume that $e = xw_i$ for some i $(1 \leq i \leq a)$, then the edge xu_i $(1 \leq i \leq a)$ will be isolated in $\langle D' \rangle$ and so D' is not a total edge dominating set of G. Therefore any proper subset of D is not a total edge dominating set of G. Hence D is a minimal total edge dominating set of G and so $\gamma_{te}^{+}(G) \geq 2a$. We show that $\gamma_{te}^{+}(G) = 2a$. Suppose that there exists a minimal total edge dominating set T of G such that $|T| \geq 2a + 1$. Then T contains at least three edges of block of $G - \{x\}$ or at least three edges of block of $G - \{y\}$. If T contains at least three edges of $G - \{x\}$, then deleting one edge of $G - \{x\}$ in T, results in T is a total edge dominating set of G, which is a contradiction. If T contains at least three edges $G - \{y\}$, then deleting one edge of $G - \{y\}$ in T, results in T is a total edge dominating set of G, which is a contradiction. Hence $\gamma_{te}^{+}(G) = 2a$.

■

Open Problem

For every pair a, b of integers with $2 \leq a < b$, does there exists a connected graph G such that $\gamma_{te}(G) = a$ and $\gamma_{te}^{+}(G) = b$?

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