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Infinite Single Integral Representation for the Polynomial Set $R_n(x_1, x_2, x_3)$

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ABSTRACT

In the present paper, an attempt has been made to express a Infinite Single Integral Representation for the polynomial set $R_n(x_1, x_2, x_3)$. Many interesting new results may be obtained as particular cases on separating the parameter.

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KEYWORD: Appell Function, Generalized Hyper geometric Polynomial, Integral Representation, Lauricella function.

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1. INTRODUCTION

We defined the generalized hypergeometric polynomial set $R_n(x_1, x_2, x_3)$ by means of generating relation,

$$\begin{aligned}
 & (1-vt)^{-\lambda} (1-\mu_1 x_2^{r_1} t^{r_1})^{-\lambda_1} \\
 & F \left[\begin{matrix} (A_p); (C_u); (E_h); (G_m) \\ \mu x_1^{r_1} t, \mu_2 x_2^{-r_2} t^{r_2}, \mu_3 x_3^{-r_3} t^{r_3} \end{matrix} \right] \\
 & \left[\begin{matrix} (B_q); (D_v); (F_k); (H_w) \end{matrix} \right] \\
 & = \sum_{n=0}^{\infty} R_{n,r;r_1;r_2;r_3;r_4}^{v;\lambda;\lambda_1;\lambda_2;\mu_1;\mu_2;\mu_3;(A_p):(C_u):(E_h):(G_m);(B_q):(D_v):(F_k):(H_w)} (x_1, x_2, x_3) t^n \dots (1.1)
 \end{aligned}$$

where $v, \lambda, \lambda_1, \lambda_2, \lambda_3$ are real and r, r_1 are non-negative integer and e_2, e_3 are natural numbers.

The left hand side of (1.1) contains the product of generalized hyper geometric function and Lauricella function in the notation of Burchanall and Chaundy¹.

The polynomial set contains number of parameters, for simplicity we shall denote

$$R_{n,r;r_1;r_2;r_3;r_4}^{v;\lambda;\lambda_1;\lambda_2;\mu_1;\mu_2;\mu_3;(A_p):(C_u):(E_h):(G_m);(B_q):(D_v):(F_k):(H_w)} (x_1, x_2, x_3)$$

By $R_n(x_1, x_2, x_3)$.

where n denotes the order of the polynomial set.

After little simplification (1.1) gives

$$\begin{aligned}
 R_n(x_1, x_2, x_3) &= \sum_{s=0}^{\lfloor \frac{n}{r} \rfloor} \sum_{s_1=0}^{\lfloor \frac{n-r}{r_1} \rfloor} \sum_{r_2=0}^{\lfloor \frac{n-r-r_1 s_1}{e_2} \rfloor} \sum_{s_3=0}^{\lfloor \frac{n-r-r_1 s_1-r_2 s_2}{e_3} \rfloor} \\
 & \times \frac{[(A_p)]_{n-r-r_1 s_1-(r_2-1)s_1-(r_3-1)s_3}}{[(B_q)]_{n-r-r_1 s_1-(r_2-1)s_2-(r_3-1)s_3}} \\
 & \times \frac{[(C_u)]_{n-r-r_1 s_1-r_2 s_2-r_3 s_3} [(E_h)]_{s_2} [(G_m)]_{s_3} (\lambda)_s (\lambda_1)_{s_1} v^s \mu_1^{s_1} \mu_2^{s_2} \mu_3^{s_3}}{[(D_v)]_{n-r-r_1 s_1-r_2 s_2-r_3 s_3} [(F_k)]_{s_2} [(H_w)]_{s_3} s! s_1! s_2! s_3!} \\
 & \times \frac{(\mu x_1^{r_4})^{n-r-r_1 s_1-r_2 s_2-r_3 s_3} x_2^{r_1 s_1+r_2 s_2}}{(n-r-r_1 s_1-r_2 s_2-r_3 s_3)! x_3^{r_3 s_3}} \dots (1.2)
 \end{aligned}$$

2. NOTATIONS

I.(i) $(n) = 1, 2, 3, \dots, n-1, n$.

(ii) $(a_p) = a_1, a_2, a_3, \dots, a_p$.

(iii) $(a_p ; i) = a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_p$.

II.(i) $[(a_p)] = a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_p$.

(ii) $[(a_p)]_n = \prod_{i=1}^p (a_i)_n = (a_1)_n (a_2)_n \dots (a_p)_n$.

III.(i) $\Delta(a, b) = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-1}{a}$.

(ii) $\Delta_k(a; b) = \left(\frac{b}{a}\right)_k \left(\frac{b+1}{a}\right)_k \dots \left(\frac{b+a-1}{a}\right)_k$
 $= \prod_{r=1}^a \left(\frac{b+r-1}{a}\right)_k$

(iii) $\Delta[m; (a_p)] = \prod_{i=1}^p \prod_{r=1}^m \left(\frac{a_i+r-1}{m}\right)_k$.

IV.(i) $\Gamma[(a_p)] = \prod_{i=1}^p \Gamma(a_i)$.

(ii) $\Gamma[(a_p); s] = \prod_{i=s+1}^p \Gamma(a_i)$.

(iii) $\Gamma\left[a + \frac{(m)}{m}\right] = \prod_{r=1}^m \Gamma\left(a + \frac{r}{m}\right)$.

(iv) $\Gamma[\Delta(a; b)] = \prod_{r=1}^a \Gamma\left(\frac{b+r-1}{a}\right)$.

V.(i) $\Gamma_*(a \pm b) = \Gamma(a+b)\Gamma(a-b)$.

(ii) $\Gamma_*(a+b) = \Gamma(a+b)\Gamma(a-b)$.

VI.(i) $M_1 = \frac{[(A_p)]_n [(C_u)]_n (\mu x_1^{r_1})^n}{[(B_q)]_n [(D_v)]_n n!}$

3. THEOREM

For $r_2 > 1$, and $r_3 > 1$, we have

$$R_n(x_1, x_2, x_3) = \frac{s^a \Gamma(d-c)\Gamma(d-a) M_1}{\Gamma(a)\Gamma(d)\Gamma(d-c-a)} \int_0^\infty e^{-ax} x^{a-1} \begin{bmatrix} c; \\ d; \end{bmatrix} a^x$$

$$\times F_{\rho+u: k: w}^{1+q+v: h: m: 1} \left[\begin{matrix} [-n: r, r_1, r_2, r_3], [1 - (B_q) - n: r, r_1, r_2 - 1, r_3 - 1] \\ \text{-----}, [1 - (A_p) - n: r, r_1, r_2 - 1, r_3 - 1] \end{matrix} \right]$$

$$[(1 - (D_v) - n): r, r_1, r_2, r_3], [(E_h): 1], [(G_m): 1], [\lambda: 1], [\lambda_1: 1], [(a): 1], [d - (-a): 1]$$

$$[(1 - (C_u) - n): r, r_1, r_2, r_3], [(F_k): 1], [(H_w): 1], [d - a; 1], \text{---}; \quad |$$

$$\frac{v(-1)^{r(p+q+u+v+1)}}{(\mu x_1^{r_4})^r}, \frac{\mu_1 x_2^{r_1} (-1)^{r_1(p+q+u+v+1)}}{(\mu x_1^{r_4})^{r_1} s},$$

$$\left. \frac{\mu_2 x_2^{r_2} (-1)^{r_2(p+q+u+v+1)+p+q}}{(\mu x_1^{r_4})^{r_2}}, \frac{\mu_3 (-1)^{r_3(p+q+u+v+1)+p+q}}{\mu (x_1^{r_4} x_3)^{r_3}} \right] dx \dots (3.1)$$

Proof : we have

$$I_5 = \int_0^\infty \rho^{-sx} x_1^{a-1} {}_1F_1 \left[\begin{matrix} c; \\ d; \end{matrix} \middle| sX_1 \right] \times \sum_{s=0}^{\lfloor \frac{n}{r} \rfloor} \sum_{s_1=0}^{\lfloor \frac{n-r}{r_1} \rfloor} \sum_{s_2=0}^{\lfloor \frac{n-r-r_1s_1}{r_2} \rfloor} \sum_{s_3=0}^{\lfloor \frac{n-r-r_1s_1-r_2s_2}{r_3} \rfloor}$$

$$\times \frac{[(A_p)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3} [(C_u)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3}}{[(B_q)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3} [(D_v)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3}}$$

$$\times \frac{[(E_h)]_{s_2} [(G_m)]_{s_3} (\lambda)_s v^s (\lambda_1)_{s_1} \mu_1^{s_1} x_2^{r_1s_1+r_2s_2} \mu_2^{s_2} \mu_3^{s_3}}{[(F_k)]_{s_2} [(H_w)]_{s_3} s! s_1! s_2! s_3! x_3^{r_3s_3}}$$

$$\times \frac{(\mu x_1^{r_4})^{n-r-r_1s_1-r_2s_2-r_3s_3} s^{s_1} (d-a)_{s_1}}{(n-r-r_1s_1-r_2s_2-r_3s_3)! (\alpha)_{s_1} (d-c-a)_{s_1}}$$

$$= \int_0^\infty \rho^{-sx} x_1^{a+s_1-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \middle| sX \right] \times \sum_{s=0}^{\lfloor \frac{n}{r} \rfloor} \sum_{s_1=0}^{\lfloor \frac{n-r}{r_1} \rfloor} \sum_{s_2=0}^{\lfloor \frac{n-r-r_1s_1}{r_2} \rfloor} \sum_{s_3=0}^{\lfloor \frac{n-r-r_1s_1-r_2s_2}{r_3} \rfloor}$$

$$\times \frac{[(A_p)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3} [(C_u)]_{n-r-r_1s_1-r_2s_2-r_3s_3} [(E_h)]_{s_2}}{[(B_q)]_{n-r-r_1s_1-(r_2-1)s_2-(r_3-1)s_3} [(D_v)]_{n-r-r_1s_1-r_2s_2-r_3s_3} [(F_k)]_{s_2}}$$

$$\times \frac{[(G_m)]_{s_3} (\lambda)_s v^s (\lambda_1)_{s_1} v^{s_1} x_2^{r_1s_1+r_2s_2} \mu_2^{s_2} \mu_3^{s_3}}{[(H_w)]_{s_3} s! s_1! s_2! s_3! x_3^{r_3s_3} (n-r-r_1s_1-r_2s_2-r_3s_3)!}$$

$$\times \frac{(\mu x_1^{r_4})^{n-r-r_1s_1-r_2s_2-r_3s_3} s^{s_1} (d-a)_{s_1} \Gamma(a+s_1) \Gamma(d) \Gamma(d-c-a-s_1)}{(a)_{s_1} (d-c-a)_{s_1} s^{a+s_1} \Gamma(d-c) \Gamma(d-a-s_1)}$$

$$= M_1 \frac{\Gamma(a) \Gamma(d) \Gamma(d-c-a)}{s^a \Gamma(d-c) \Gamma(d-a)} \times \sum_{s=0}^{\lfloor \frac{n}{r} \rfloor} \sum_{s_1=0}^{\lfloor \frac{n-r}{r_1} \rfloor} \sum_{s_2=0}^{\lfloor \frac{n-r-r_1s_1}{r_2} \rfloor} \sum_{s_3=0}^{\lfloor \frac{n-r-r_1s_1-r_2s_2}{r_3} \rfloor}$$

$$\times \frac{[1-(B_q)-n]_{r+r_1s_1+(r_2-1)s_2+(r_3-1)s_3} [1-(D_v)-n]_{r+r_1s_1+r_2s_2+r_3s_3}}{[1-(A_p)-n]_{r+r_1s_1+(r_2-1)s_2+(r_3-1)s_3} [1-(C_u)-n]_{r+r_1s_1+r_2s_2+r_3s_3}}$$

$$\begin{aligned} & \times \frac{[(E_h)]_{s_2} [(G_m)]_{s_3} (\lambda)_s v^s (-1)^{r(p+q+u+v+1)} \mu_1^{s_1} (-1)^{r_1(p+q+u+v+1)s_1} x_2^{r_1 s_1} (\lambda_1)_{s_1}}{[(F_k)]_{s_2} [(H_w)]_{s_3} s! (\mu x_1^{r_4})^{r+r_1 s_1+r_2 s_2+r_3 s_3} s_1!} \\ & \times \frac{\mu_2^{s_2} x_2^{r_2 s_2} (-1)^{r_2(p+q+u+v+1)+p+q} s_2 \mu_3^{s_3}}{s_2! s^{s_1}} \dots (3.2) \end{aligned}$$

The single terminating factor $(-n)_{r+r_1 s_1+r_2 s_2+r_3 s_3}$ makes all summation in (3.2), runs upto ∞ .

$$\begin{aligned} & = \frac{M_1 \Gamma(a)\Gamma(d)\Gamma(d-c-a)}{s^a \Gamma(d-c) \Gamma(d-a)} \sum_{s=0}^{\infty} \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \sum_{s_3=0}^{\infty} \\ & \times \frac{[1-(B_q)-n]_{r+r_1 s_1+(r_2-1)s_2+(r_3-1)s_3} [1-(D_v)-n]_{r+r_1 s_1+r_2 s_2+r_3 s_3}}{[1-(A_p)-n]_{r+r_1 s_1+(r_2-1)s_2+(r_3-1)s_3} [1-(C_u)-n]_{r+r_1 s_1+r_2 s_2+r_3 s_3}} \\ & \times \frac{[(E_h)]_{s_2} [(G_m)]_{s_3} (\lambda)_s v^s (-1)^{r(p+q+u+v+1)} (\lambda_1)_{s_1} \mu_1^{s_1} x_2^{r_1 s_1}}{[(F_k)]_{s_2} [(H_w)]_{s_3} s! (\mu x_1^{r_4})^r s^{s_1} s_1! (\mu x_1^{r_4})^{r_1 s_1}} \\ & \times \frac{(-1)^{r_1(p+q+u+v+1)s_1} (-1)^{r_2(p+q+u+v+1)+p+q} s_2 x_2^{r_2 s_2} \mu_2^{s_2}}{(\mu x_1^{r_4})^{r_2 s_2}} \\ & \times \frac{\mu_3^{s_3} (-1)^{r_3(p+q+u+v+1)+p+q} s_3}{(\mu x_1^{r_4} x_3)^{r_3 s_3}} \\ & = M_1 \frac{\Gamma(a)\Gamma(d)\Gamma(d-c-a)}{s^a \Gamma(d-c) \Gamma(d-a)} R_n(x_1, x_2, x_3) \end{aligned}$$

On using²

$$\int_0^{\infty} e^{-st} t^{a-1} {}_1F_1 \left[\begin{matrix} c; \\ d; \end{matrix} \middle| st \right] dt = \frac{\Gamma(a)\Gamma(d)\Gamma(d-c-a)}{s^a \Gamma(d-c) \Gamma(d-a)}$$

Where $R_e(a), R_e(s) > 0, R_e(d) > R_e(c + a)$.

PARTICULAR CASES OF (3.1)

On specializing the values of the parameters involved in Lauricella form, a number of known and unknown polynomials can be obtained as the particular cases of polynomial set $R_n(x_1, x_2, x_3)$. Some of them, which are well known, are listed below.

I. Gegenbauer Polynomials

On taking $p = 0 = q = u = h = s; v = 0 = k = r = r_2 = r_4 = \lambda = v = x_2;$

$$D_1 = \lambda + \frac{1}{2} = F_1; \mu = \frac{1}{2} = \mu_2$$

andwriting $\frac{x+1}{x-1}$ for x_1 in (3.1), we get

$$C_n^\lambda(x) = \frac{s^a \Gamma(d-a)\Gamma(d-c)(2\lambda)_n}{\Gamma(a)\Gamma(d)\Gamma(d-c-a)n!} \left(\frac{x-1}{2}\right)^n \int_0^\infty e^{-sx} t^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \middle| sx \right] \\ \times F \left[\begin{matrix} -n, \frac{1}{2} - \lambda - n(a:1), (d-c-a:1); \\ \lambda + \frac{1}{2}, (d-a:1); \end{matrix} \middle| \frac{x-1}{x+1} \times \frac{1}{s} \right] dx$$

where $C_n^\lambda(x)$ are the Gegenbauer Polynomials.

II. Gegenbauer Polynomials

On making the substitution $p = 0 = q = u = s; u = v = 1 = r = r_4 = x_2 = \lambda = v = \mu = \mu_2;$

$F_1 = \lambda + \frac{1}{2}, r_2 = 2$ and writing $\frac{x}{\sqrt{x^2-1}}$ for x_1 in (3.1), we get

$$C_n^\lambda(x) = \frac{s^a \Gamma(d-a)\Gamma(d-c)(2\lambda)_n}{n! \Gamma(a)\Gamma(d)\Gamma(d-a-c)} \int_0^\infty e^{-sx} t^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \middle| sx \right] \\ \times F \left[\begin{matrix} \frac{-n}{2}, \frac{-n}{2} + \frac{1}{2}, (s:1), (d-c-a:1); \\ \lambda + \frac{1}{2}, (d-a:1); \end{matrix} \middle| \frac{x^2-1}{sx^2} \right] dx$$

where $C_n^\lambda(x)$ are the Gegenbauer Polynomials.

III. Legendre Polynomials

If we put $p = 0 = q = u = v = m; w = 1 = r = r_4 = v = \lambda = \mu = \mu_3 = r_3; H_1 = 1, r_3 = 2$

and $\frac{x}{\sqrt{x^2-1}}$ for x_1 in (3.1), we get

$$P_n(x) = \frac{s^a \Gamma(d-a)\Gamma(d-c)x^n}{n! \Gamma(a)\Gamma(d)\Gamma(d-a-c)} \int_0^\infty e^{-sx} x^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \middle| sx \right] \\ \times F \left[\begin{matrix} \frac{-n}{2}, \frac{-n}{2} + \frac{1}{2}, (s:1), (d-c-a:1); \\ 1, (d-a:1); \end{matrix} \middle| \frac{x^2-1}{sx^2} \right] dx$$

where $P_n(x)$ are the Legendre Polynomials

IV. Panda, Rekha Polynomials

On making the substitution $\rho = 0 = q = u = v$; $r = 1 = r_4 = v = \lambda = x_2$; $\mu_3 = \mu = v$, $r_3 = m$, $x_1 = x$ and replacing (G_m) by (a_r) and (H_w) by (b_s) in (3.1), we arrive at

$$A_n(x) = \frac{s^a \Gamma(d-a)\Gamma(d-c)(vx)^n}{n! \Gamma(a) \Gamma(d) \Gamma(d-a-c)} \int_0^\infty e^{-sx} x^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \begin{matrix} \\ sx \end{matrix} \right] \times F \left[\begin{matrix} \Delta(m; -n)(a_r), (s; 1), (d-c-a; 1); \\ \frac{\mu}{s} \left(\frac{-m}{vx} \right)^m \\ (b_s), (d-a; 1); \end{matrix} \right] dx$$

where $A_n(x)$ are the generalized polynomials defined by Panda³

V. Brafman Polynomials:

On putting $\rho = 0 = q = u = v = s$; $r = 1 = r_4 = \lambda = v = \mu = \mu_3 = x_1 = x_2$; $r_3 = \rho = \mu$; $\mu_3 = x$ in (3.1) and $G_m = \alpha_u$; $H_n = \beta_v$; we get

$$B_n^\rho(x) = \frac{s^a \Gamma(d-a)\Gamma(d-c)(-p)^n}{n! \Gamma(a) \Gamma(d) \Gamma(d-a-c)} \int_0^\infty e^{-sx} x^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \begin{matrix} \\ sx \end{matrix} \right] \times F \left[\begin{matrix} \Delta(\rho; -n), \alpha_1, \alpha_2, \dots, \alpha_u, (s; 1), (d-c-a; 1); \\ \frac{x}{s} \\ (d-a; 1), \beta_1, \beta_2, \dots, \beta_v; \end{matrix} \right] dx$$

where $B_n^\rho(x)$ are the generalization of Hermite polynomials by Brafman⁴

VI. Lommel Polynomials

$q = 0 = v = m = s$; $p = 1 = u = w = \lambda = v = x = r_4$; $\mu_3 = -1$, $r_3 = 2 = \mu$; $A_1 = 1$, $C_1 = v$, $H_1 = v$ and z for x_1 in (3.1), we achieve

$$R_{n,v} \left(\frac{1}{z} \right) = \frac{s^a \Gamma(d-a)\Gamma(d-c)(v)_n (2z)^n}{n! \Gamma(a) \Gamma(d) \Gamma(d-a-c)} \int_0^\infty e^{-sx} x^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \begin{matrix} \\ sx \end{matrix} \right] \times F \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, (s; 1), (d-c-a; 1); \\ \frac{-1}{sz^2} \\ v, -n, 1-v-n, (d-a; 1); \end{matrix} \right] dx$$

where $R_{n,v} \left(\frac{1}{z} \right)$ are the Lommel Polynomials⁵

VII. Khanna, I.K Polynomials

On taking $p = 0 = q = s; u = 1 = v = m = w = \lambda = \nu = r = r_4 = x_2 = y, r_3 = m, \mu_3 = \mu; \mu = \nu$ and instead $(D_U) = (\beta_q); x_1 = x (C_U) = (\alpha_p), (G_M) = (a_r)$ and $(H_W) = (b_s)$ in (3.1), we arrive at

$$B_n(x, y) = \frac{s^a \Gamma(d-a)\Gamma(d-c) [(\alpha_p)]_n (vx)^n}{n! \Gamma(a) \Gamma(d) \Gamma(d-a-c) [(\beta_q)]_n} \int_0^\infty e^{-sx} x^{a-1} F \left[\begin{matrix} c; \\ d; \end{matrix} \begin{matrix} \\ sx \end{matrix} \right] \times F \left[\begin{matrix} \Delta(m; -n), \Delta(m, 1 - (\beta_q) - n), (a_r), \\ (s; 1), (d - c - a : 1); \\ \frac{\mu(-m)^{m(q-p+1)}}{s(vxy)^m} \\ \Delta(m; 1 - (\alpha_p) - n), (b_s), (d - a : 1); \end{matrix} dx \right]$$

where $B_n(x, y)$ are the Polynomials defined by Khanna I.K.⁶

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