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Weyl Fractional Derivative of Multivariable Polynomials and I- Function

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ABSTRACT

In this research work, we establish a theorem on Weyl fractional derivative of the product of multivariable polynomials and I- function in the literature of special functions. The results are obtain in the compact form containing the multivariable polynomials. Some special cases of our theorem have been discussed.

KEYWORDS- Weyl fractional derivative operator, multivariable polynomials and I-function.

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INTRODUCTION

Recently, for modeling of relevant systems in various fields of sciences and engineering, such as fluid flow, diffusion, relaxation, oscillation, anomalous diffusion, polymer physics, chemical physics, propagation of seismic waves, etc. see, Glöckle and Nonnenmacher¹, Mainardi^{2, 3}, Miller and Ross⁴, Kilbas et al.⁵ and others.

The I-function of the one variable is defined by Saxena⁶ and we will represent here in the following manner:

$$I[z] = I_{p_i, q_i; r}^{m, n}[z] = I_{p_i, q_i; r}^{m, n}\left[z \left| \begin{array}{c} \dots, \dots \\ \dots, \dots \end{array} \right. \right] = I_{p_i, q_i; r}^{m, n}\left[z \left| \begin{array}{l} (a_j, e_j)_{1, n}; \quad (a_{ji}, e_{ji})_{n+1, pi} \\ (b_j, f_j)_{1, m}; \quad (b_{ji}, f_{ji})_{m+1, qi} \end{array} \right. \right] \quad (1.1)$$

$$= \frac{1}{2\pi i} \int_L \theta(s) z^s ds, \quad (1.2)$$

where $i = \sqrt{(-1)}$, $z (\neq 0)$ is a complex variable and (1.2) $z^s = \exp[s \{\log |z| + i \arg z\}]$. In which $\log |z|$ represent the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principle value. An empty product is interpreted as unity. Also,

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s)}{\sum_{i=1}^r \left[\prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + f_{ji} s) \prod_{j=n+1}^p \Gamma(a_{ji} - e_{ji} s) \right]} \quad (1.3)$$

m, n, p_i and $q_i \forall i \in (1, \dots, r)$ are non-negative integers satisfying $0 \leq n \leq p_i$, $0 \leq m \leq q_i; \forall i \in \{1, \dots, r\}$, $e_{ji}, (j = 1, \dots, p_i; i = 1, \dots, r)$ and $f_{ji}, (j = 1, \dots, q_i; i = 1, \dots, r)$ are assumed to be positive quantities for standardization purpose. Also $a_{ji}, (j = 1, \dots, p_i; i = 1, \dots, r)$ and $b_{ji}, (j = 1, \dots, q_i; i = 1, \dots, r)$ are complex numbers such that none of the points.

$$S = \{(b_n + v) | f_h\}, h = 1, \dots, m; v = 0, 1, 2, \dots, \quad (1.4)$$

Which are the poles of $\Gamma(b_h - f_h s), h = 1, \dots, m$ and the points

$$S = \{(a_l - n - 1) | e_l\}, l = 1, \dots, n; \eta = 0, 1, 2, \dots, \quad (1.5)$$

Which poles are of $\Gamma(1 - a_l + e_l s)$ coincide with one another, i.e. with

$$e_l(b_n + v) \neq b_n(a_l - \eta - 1), \quad (1.6)$$

for $v, \eta = 0, 1, 2, \dots; h = 1, \dots, m; l = 1, \dots, n$.

Further, the contour L runs from $-i_\infty$ to $+i_\infty$. Such that the poles of $\Gamma(b_h - s)$, $h = 1, \dots, m$; lie to the right of L and the poles $\Gamma(1 - a_l + e_l s)$, $l = 1, \dots, n$ lie to the left of L. The integral converges, if $|\arg z| < \frac{1}{2}B\pi$, $B > 0$, $A \leq 0$, where

$$A = \sum_{j=1}^{p_i} e_{ji} - \sum_{j=1}^{q_i} f_{ji} \quad \text{and} \quad (1.7)$$

$$B = \sum_{j=1}^n e_j - \sum_{j=n+1}^{p_i} e_{ji} + \sum_{j=1}^m f_j - \sum_{j=m+1}^{q_i} f_{ji}, \quad \forall i \in (1, \dots, r) \quad (1.8)$$

Let A denote a class of good functions. By good function f, we mean Miller⁷ a function which is everywhere differentiable any number of times and if it is all of its derivatives are $O(x^{-v})$, for all v as x increases without limit. We define the Weyl fractional derivatives of a function g(z) as follows:-

Let $g \in A$, then for $q < 0$,

$${}_zW_\infty^q g(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1} g(u) du. \quad (1.9)$$

For $q \geq 0$

$${}_zW_\infty^q g(z) = \frac{d^n}{dz^n} ({}_zW_\infty^{q-n} g(z)), \quad (1.10)$$

Where n being positive integer, such that $n > q$.

The general class of multivariable polynomials is defined by Srivastava and Garg⁸.

$$S_L^{h_1, \dots, h_r} [x_1, \dots, x_r] = \sum_{k_1, \dots, k_r=0}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!}, \quad (1.11)$$

Where h_1, \dots, h_r are positive integers and the co-efficient $A(L; k_1, \dots, k_r)$, ($L; h_i \in N$; $i = 1, \dots, r$) are arbitrary constant, real or complex.

Evidently the case $r = 1$ of the polynomials(1.11).

Would correspond the polynomials given by Srivastava⁹.

$$S_L^h[x] = \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} x^k \quad \{L \in N = (0,1,2,\dots)\}, \quad (1.12)$$

where h is arbitrary positive integers and the co-efficient $A_{L,k}$ ($L, k \geq 0$) are arbitrary constant, real or complex.

2. MATHEMATICAL PRE-REQUISITES - To establish the main result, we need the following integral of the H-function by Saigo¹⁰.

$$\begin{aligned} & \int_x^\infty t^{\rho-1} (t-x)^{\sigma-1} I_{p_i, q_i; r}^{m, n} \left[z t^\mu (t-x)^\nu \left| \begin{matrix} (a_j, e_j)_{1,n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1,m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right] dt \\ &= x^{\rho+\sigma-1} I_{p_i+2, q_i+1; r}^{m+1, n+1} \left[z x^{\mu+\nu} \left| \begin{matrix} (1-\sigma, \nu), (a_j, e_j)_{1,n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho, \mu) \\ (1-\rho-\sigma, \mu+\nu), (b_j, f_j)_{1,m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{matrix} \right. \right], \end{aligned} \quad (2.1)$$

Where

(i) ρ, σ are complex numbers and μ, ν are positive real numbers,

(ii) $|\arg z| < \frac{1}{2} A\pi$, A defined as $A = \sum_{j=1}^{p_i} e_{ji} - \sum_{j=1}^{q_i} f_{ji}$,

(iii) $\min \left[\operatorname{Re} \left(\frac{1-\rho-\sigma}{\mu-\nu} \right), \min 1 \leq j \leq m \left[\operatorname{Re} \left(\frac{b_j}{f_j} \right) \right] \right] > \max \left[-\operatorname{Re} \left(\frac{\sigma}{\nu} \right), \max 1 \leq j \leq N \left[\operatorname{Re} \left(\frac{a_j - 1}{e_j} \right) \right] \right]$.

3. WEYL FRACTIONAL DERIVATIVE OF MULTIVARIABLE POLYNOMIALS AND I- FUNCTION-

THEOREM. Let m, n, p_i and q_i be non-negative integers such that $0 \leq n \leq p_i, 0 \leq m \leq q_i$ and

$\sum_{j=1}^n e_j - \sum_{j=n+1}^{p_i} e_{ji} + \sum_{j=1}^m f_j - \sum_{j=m+1}^{q_i} f_{ji} > 0$ together with the set of conditions (i) – (iii) given with equation

(2.1). Then, for all value of q ,

$${}_z W_\infty^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^{h_1, \dots, h_r} [c_1 x^{\delta_1}, \dots, c_r x^{\delta_r}] \right\}$$

$$\begin{aligned}
 & \times I_{p_i, q_i; r}^{m, n} \left[yx^{\mu} (x-z)^v \left| \begin{array}{l} (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right] \\
 = & \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-\sum_{i=1}^r k_i \delta_i - 2} \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \\
 & \times I_{p_{i+2}, q_{i+1}; r}^{m+1, n+1} \left[yz^{\mu+v} \left| \begin{array}{l} (2-\sigma+q, v), (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+v), (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right]. \quad (3.1)
 \end{aligned}$$

PROOF - Taking left hand side of equation (3.1) and using equation (1.11), we get

$$\begin{aligned}
 {}_z W_{\infty}^q & \left\{ x^{\rho+\sum_{i=1}^r k_i \delta_i - 1} (z-x)^{\sigma-1} \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \right. \\
 & \left. \times I_{p_i, q_i; r}^{m, n} \left[yx^{\mu} (x-z)^v \left| \begin{array}{l} (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i} \\ (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right] \right\}, \quad (3.2)
 \end{aligned}$$

Now using equation (1.9) and definition of I- function, easily we can find the proof of equation (3.1).

For $q \geq 0$ invoking the definition (1.10) the relation (3.2) further reduces to

$$\begin{aligned}
 & \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \frac{(-1)^{q+r+\sigma-1}}{\Gamma(r-q)} \frac{d^r}{dz^r} \{ z^{\rho-\sigma-q+r-\sum_{i=1}^r k_i \delta_i - 2} \\
 & I_{p_{i+2}, q_{i+1}; r}^{m+1, n+1} \left[yz^{\mu+v} \left| \begin{array}{l} (2-\sigma-r+q, v), (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+v), (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right] \}
 \end{aligned}$$

In replacing of $(q - r)$ by q , we may obtain again

$$\begin{aligned}
 & \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-\sum_{i=1}^r k_i \delta_i - 2} \sum_{\substack{k_1, \dots, k_r=0 \\ h_1 k_1 + \dots + h_r k_r}}^{h_1 k_1 + \dots + h_r k_r \leq L} (-L)_{h_1 k_1 + \dots + h_r k_r} A(L; k_1, \dots, k_r) \frac{c_1^{k_1}}{k_1!} \dots \frac{c_r^{k_r}}{k_r!} \\
 & \times I_{p_{i+2}, q_{i+1}; r}^{m+1, n+1} \left[yz^{\mu+v} \left| \begin{array}{l} (2-\sigma+q, v), (a_j, e_j)_{1, n}; (a_{ji}, e_{ji})_{n+1, p_i}, (1-\rho-\sum_{i=1}^r k_i \delta_i, \mu) \\ (2+q-\rho-\sigma-\sum_{i=1}^r k_i \delta_i, \mu+v), (b_j, f_j)_{1, m}; (b_{ji}, f_{ji})_{m+1, q_i} \end{array} \right. \right] \quad (3.3)
 \end{aligned}$$

4. SPECIAL CASE - If we put $r=1$ in the general call of multivariable polynomials given by Srivastava and Garg⁸ reduces to the polynomials given by Srivastava(1972) and I- function reduces into Fax's H – function as follows :

$$\begin{aligned} {}_zW_{\infty}^q & \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{\nu} \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\ & = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+2,q+1}^{m+1,n+1} \left[yz^{\mu+\nu} \left| \begin{matrix} (2-\sigma+q, \nu), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu+\nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right]. \quad (4.1) \end{aligned}$$

Replacing ν by $-\nu$ equation (4.1) correspond to the following result according as $\mu > \nu$, $\mu < \nu$ and $\mu = \nu$, i.e. for $\mu > \nu$

$$\begin{aligned} {}_zW_{\infty}^q & \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{-\nu} \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\ & = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+1,q+2}^{m+2,n} \left[yz^{\mu-\nu} \left| \begin{matrix} (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu-\nu), (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right] \quad (4.2) \end{aligned}$$

For $\mu < \nu$

$$\begin{aligned} {}_zW_{\infty}^q & \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{-\nu} \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\ & = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+2,q+1}^{m+1,n+1} \left[yz^{\mu-\nu} \left| \begin{matrix} (\rho+\sigma-q+k-1, \nu-\mu), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right] \quad (4.3) \end{aligned}$$

and $\mu = \nu$

$$\begin{aligned} {}_zW_{\infty}^q & \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h [x^k] \times H_{p,q}^{m,n} \left[yx^{\mu} (x-z)^{-\nu} \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\ & = \frac{(-1)^{q+\sigma-1} \Gamma(2-q-\rho-k)}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+1,q+1}^{m+1,n} \left[y \left| \begin{matrix} (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (\sigma-q-1, \nu), (b_j, f_j)_{1,q} \end{matrix} \right. \right] \quad (4.4) \end{aligned}$$

Finally writing $-\mu$ instead of μ , equation (4.1) yields the following results according as $\mu > \nu$, $\mu < \nu$ and $\mu = \nu$ respectively.

For $\mu > \nu$

$$\begin{aligned}
 & {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h \left[x^k \right] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^v \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \\
 & = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+2,q+1}^{m,n+2} \left[yz^{-\mu+v} \middle| \begin{matrix} (q-\rho-\sigma-k+1, \mu-v), (2+q-\sigma, v), (a_j, e_j)_{1,p} \\ (b_j, f_j)_{1,q}, (\rho+k, \mu) \end{matrix} \right] \quad (4.5)
 \end{aligned}$$

for $\mu < v$

$$\begin{aligned}
 & {}_zW_{\infty}^q \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h \left[x^k \right] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^v \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\
 & = \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \times H_{p+2,q+1}^{m+1,n+1} \left[yz^{-\mu+v} \middle| \begin{matrix} (2-\sigma+q, v), (a_j, e_j)_{1,p} \\ (2+q-\rho-\sigma-k, v-\mu), (b_j, f_j)_{1,q}, (\rho+k, \mu) \end{matrix} \right] \quad (4.6)
 \end{aligned}$$

and for $\mu = v$

$$\begin{aligned}
 & {}_zW_{\infty}^q \left\{ {}_zW_{\infty}^q \left\{ x^{\rho-1} (z-x)^{\sigma-1} S_L^h \left[x^k \right] \times H_{p,q}^{m,n} \left[yx^{-\mu} (x-z)^v \middle| \begin{matrix} (a_1, e_1)_{1,p} \\ (b_1, f_1)_{1,q} \end{matrix} \right] \right\} \right\} \\
 & = \frac{\Gamma(2-\rho-k-\sigma+q)}{\Gamma(-q)} \frac{(-1)^{q+\sigma-1}}{} z^{\rho+\sigma-q-k-2} \sum_{k=0}^{[L,h]} \frac{(-L)_{hk}}{k!} A_{L,k} \\
 & \times H_{p+2,q+1}^{m+1,n+1} \left[yz^{\mu+v} \middle| \begin{matrix} (2-\sigma+q, v), (a_j, e_j)_{1,p}, (1-\rho-k, \mu) \\ (2+q-\rho-\sigma-k, \mu+v), (b_j, f_j)_{1,q} \end{matrix} \right]. \quad (4.7)
 \end{aligned}$$

DISCUSSION

We have obtained the results namely theorem and special cases which satisfied all the conditions mention in the statement.

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