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A New Tactic to Solve the Two Person Zero Sum Game Without Saddle Point

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ABSTRACT

Aim of this paper is to investigate a new approach to solve two person zero sum game without saddle point. In this case the best strategies are mixed strategies and the mathematical models of a two person zero sum game where in decision makers have well defined set of strategies are presented. However, as usually happens in real practical problems, these remains some lack of precision of knowledge associated with the pay-off matrix. So many theoretic approaches have already been introduced by some authors in the game theory in recent past. The theoretical support provided by the use of probability approach indeed is a very useful tool to model the problem appropriately. We therefore use new approach to circumvent the imprecision involved in our problem. This type of problems solved easily by several primal-dual methods like Arithmetic method, Algebraic method, Matrix method, Calculus method etc. the new method proposes momentous advantages over similar methods.

KEYWORDS: Game theory, mixed strategy, two person zero sum game, saddle point.

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INTRODUCTION

Game theory aims to understand situations in which decision-makers interact. Games in the everyday sense "A competitive activity in which players contend with each other according to a set of rules". In the words of a dictionary- is an example of such a situation, but the scope of the game theory is very much larger and a little space is derived to games in the everyday sense. The main focus is the use of game theory to illuminate economic, political and biological phenomena.

Game theory is a mathematical theory that deals with the general features of competitive situations. The game theory deals with making optimal decisions involving two or more participants, with the participants themselves being treated as decision makers, is called a game. It is well known that the decision process in any type of multi-criteria decision making problem is crowded with several kinds of internal complexities such as ambiguity impression etc. Here a competitive system which possess both the factors, ambiguity as well as impressing, which naturally will influence the judgement of decision makers.

In a competitive environment, the competing parties devise different strategies for success. Amongst the various possible alternatives as strategies, the best is selected for the purpose of making an effective decision. In a business situation two competitors are involved; the decision maker has two studies the move of his competitors. Therefore, a game between the two concern competitors arises as a result of the actions and interactions that are involved. Business firms competing with one another are of various categories. There are those firms that try to find the best amongst the various media of advertising such as radio, newspaper, T.V., etc. Also, business firms dealing with more or less similar products try to explore various strategies for capturing each other's markets and there by attract customers.

In other words game theory is a mathematical theory that deals with the general features of competitive situations. Where the mutual conflicting situation is not solved by individuals or organizations, game theory solves it by presenting a feasible solution.

Some Game theoretic ideas can be traced to the 18th century, but the major development of the theory began in the 1920's with the work of the mathematician Emile Borel[1] and the polymath John Von Neumann[2]. A decision event in the development of the theory was published in 1944 in the book "Theory of Games and Economic Behaviour" by Von Neumann and Oskar Morgenstern, which established the foundations of the field. John F. Nash[3] developed a key concept and initiated the game theoretic study of bargaining. Soon after Nash's work, game theoretic model began to be used in economic theory and political science and psychologist began studying how human subjects behave in experimental games. In the 1970's game theory was first use as a tool in evolutionary biology. Subsequently, game theoretic methods has come to dominate microeconomic theory and are used also in many other fields of economics and wide range of other social and behaviour sciences. Nobel Prize in economic sciences was awarded to John C.Harsanyi[4], John F.Nash[3] and R.C. Selten[5] for their effective work in the field of game theory.

The approach to competitive problem developed by John Von Neumann known as father of game theory, utilises the minimal principle which involve the fundamental idea of minimization of maximum loss or the maximum of the minimum gain. Also, several researcher (Aumann and Peleg[6], Dantzig[7], Dresher et al.[8], Dresher and Shapley[9], Hart and Schmeidler[10], Kohlberg[11], Kuhn[12], Kuhn and Tucker[13], O'Neill[14], Scarf[15] and Wolfe[16]) also contribute their work in the field of Game Theory. The game theory is capable of analysing very simple competitive situations; it cannot handle all the competitive situations that may arise.

PAY-OFF MATRIX

The pay-offs (quantitative measures in terms of gains or losses, when players select their particular strategies (course of action), can be represented in the form of the loss of other and vice versa. In other words, one player's payoff table would contain the same amount in payoff table of other player, with the sign changed. Thus, it is sufficient to construct a payoff table only for one of the players.

Player A has m strategies represented by the letters $A_1, A_2, \ldots, A_m$ and player B has n strategies represented by the letters $B_1, B_2, \ldots. B_n$. The numbers m and n need not be equal. The total number of possible outcomes is therefore m × n. Here, it is assumed that each player not only knows his own list of possible course of action but also of his opponent. For convenience, it is assumed that player A is always a gainer whereas player B a loser. Let a_{ij} be the payoff that player A gains from player B. if player A chooses strategy i and player B chooses strategy j. then the payoff matrix is shown in the Table 1.

Table 1: Payoff matrix

Player A's strategies	Player B's strategies				
	B_1	B_2	. . . . .	. . . . .	B_n
A_1	A_{11}	A_{12}	....	...	A_{1n}
A_2	A_{12}	A_{22}	...	...	A_{2n}
. . .	. . .	. . .	... ...	... ...	
A_3	A_{m1}	A_{m2}	...	...	A_{mn}

By convention, the rows of the payoff matrix denote player A strategies and the column denote player B strategies. Since player A is assumed to always be the gainer, he therefore wishes to gain as large a payoff a_{ij} as possible, player B on the other hand would do his best to reach as small as a value of a_{ij} as possible. Of course, the gain to player B and loss to A must be $-a_{ij}$.

The strategy of a player is the pre-determined rule by which a player decides his course of action from his own list of courses of action during the game. There are following two types of strategies:

(a) ***Pure Strategies***: A pure strategies is a decision, in advance of all place, always to choose a particular course of action. Pure strategies may be identified by a number representing the course of action chosen and the game has a saddle point.

(b) ***Mixed strategies***: A mixed strategy is a decision, in advance of all places, to choose a course of action for each play in accordance with some particular probability distribution and no saddle point exists in the game.

MIXED STRATEGIES IN MATRIX GAMES

Let $X = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, ..., \mathrm{x}_m)$ and $Y = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, ..., \mathrm{y}_n)$ be the mixed strategies of the row player and the column player, respectively. Note that a_{ij} is player 1's payoff when the row player chooses rows I and column player chooses column j with probability 1. The corresponding payoff for the column player is $-a_{ij}$. The expected payoff to the row player with the above mixed strategies x and y is given by:

$$= u_1(\mathrm{x}, \mathrm{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = \mathrm{xAy} \text{ ; where } x = (\mathrm{x}_1, ..., \mathrm{x}_n); \mathrm{y} = (\mathrm{y}_1, ..., \mathrm{y}_n)^T; A = [\mathrm{a}_{ij}] \quad \text{.......... (1)}$$

The expected payoff to column player $= -xAy.$ when the row player plays x, he assures himself of an expected payoff

$$= \min_{y \in \Delta(\mathrm{s}_2)} xAy \quad \text{.......... (2)}$$

The row player should therefore look for a mixed strategy x that maximize the above. i.e. an x such that

$$= \max_{x \in \Delta(\mathrm{s}_1)} \min_{y \in \Delta(\mathrm{s}_2)} xAy \quad \text{.......... (3)}$$

In other words, an optimal strategy for row player is to do max minimization. Note that the row player chooses a mixed strategy that is best for her on the assumption that whatever she does, the column player will choose an action that will hurt her (row player) as much as possible. This is a direct consequence of rationality and the fact that the payoff for each player is the negative of the other player's payoff.

Similarly, when the column player plays y, he assures himself of a payoff

$$= \min_{x \in \Delta(\mathrm{s}_1)} - xAy = - \max_{x \in \Delta(\mathrm{s}_1)} xAy \quad \text{.......... (4)}$$

That is, he assures himself of losing no more than

$$= \max_{x \in \Delta(\mathrm{s}_1)} xAy \quad \text{.......... (5)}$$

The column player's optimal strategy should be to minimize loss

$$= \min_{y\in\Delta(s_2)} \max_{x\in\Delta(s_1)} xAy \qquad \text{………. (6)}$$

This is called min maximization.

Nash Equilibrium:

The Nash equilibrium concept is motivated by the idea that a theory of rational decision-making should not be a self-destroying prediction that creates an incentive to deviate for those who believe it. A strategy profile $x \in \Theta$ is Nash equilibrium if it is a best reply to itself, namely, if: $u_i(x_i, x_{-i}) \geq u_i(z_i, x_{-i})$ for all $i = 1 \ldots n$

and all strategies $z_i \in \Delta i$. If strict inequalities hold for all $z_i \neq x_i$, then x is said to be a strict Nash equilibrium.

Games without Saddle Point (Mixed Strategies)

There are some games for which no saddle point exists. In such cases both the players must determine an optimal combination of strategies to find a saddle (equilibrium) point. The optimal strategy combination for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available chooses of strategy.

The value of game obtained by the use of mixed strategies represent least pay-off which player A can expect to win and the least which player B can lose. The expected pay-off to a player in a game with arbitrary pay-off matrix [a_{ij}] of order m × n is defined as

$$E(\mathrm{m},\mathrm{n}) = \sum_{i=1}^{k}\sum_{j=1}^{k} m_i a_{ij} n_j \qquad \text{………. (7)}$$

$= M^T AN$ (in matrix notation),

where $M = (\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3, \ldots\ldots, \mathrm{m}_p)$ and $N = (\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3, \ldots\ldots, \mathrm{n}_q)$ denote the strategies for player A and B, respectively. Also $\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3, \ldots\ldots, \mathrm{m}_p = 1$ and $\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3, \ldots\ldots, \mathrm{n}_q = 1$. Player A chooses a particular strategy with particular probability; this can also be interpreted as the relative frequency with which a strategy is chosen from the number of strategies available to a player in a particular game.

Table 2: Number of strategies available to the players

		Player B		
		I	II	
Player A	I	p_{11}	p_{12}	m
	II	p_{21}	p_{22}	1-m
		n	1-n	

Two Person Zero Sum (or Rectangular) Games

A game with only two players in which the gains of one player are the losses of another players, is called a two person zero sum game. In other words the games in which the algebraic sum of gains and losses of all the players is zero are called zero sums games. Two person, zero sum games are also called rectangular games because these are usually represented by a payoff matrix in rectangular form.

Following are the basic assumptions of the two person zero sum games:

(a) Each player has available to him a finite number of possible courses of action. The list may not be same for each player.
(b) Player A attempts to maximise gains and player B minimize losses.
(c) The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
(d) Both the players know not only the possible pay-offs to themselves but also of each other.

Various methods discussed by different authors to find value of the game under decision-making environment of certainty are as follows.

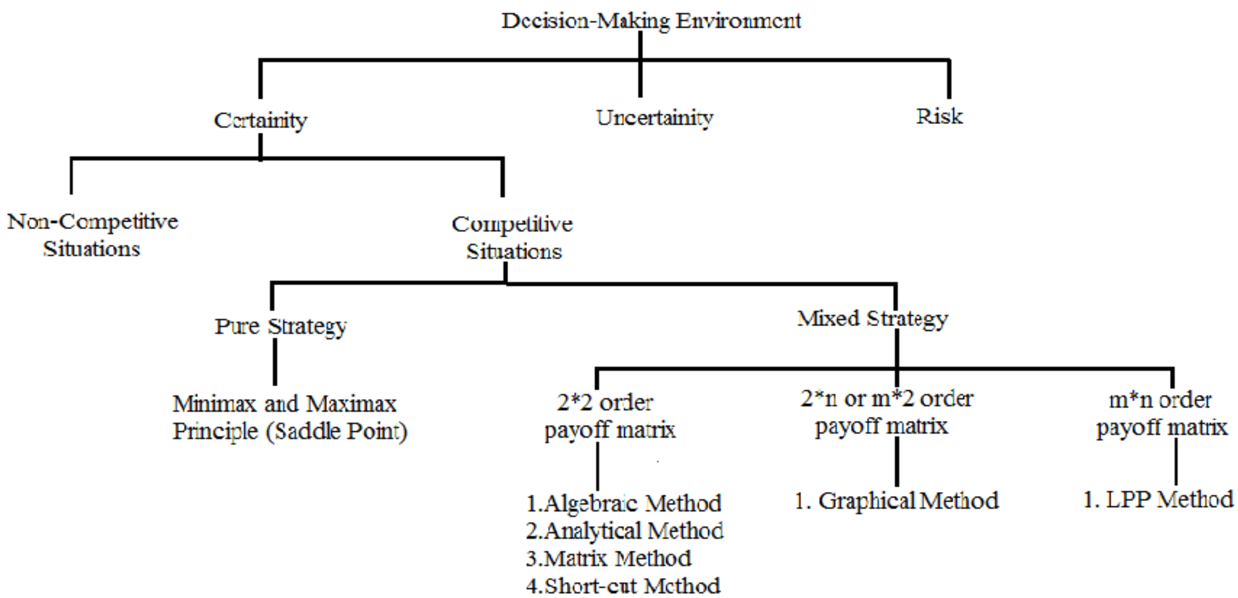

Figure1: Types of methods

MATHEMATICAL FORMULATION AND ANALYSIS

In a two-person zero sum game, the resulting gain can easily be represented in the form of a matrix, called the payoff matrix or gain matrix. Thus, a payoff matrix is a table which shows how payments should be made at the end of a play or game.

TWO PERSON ZERO SUM GAME IN TERMS OF MIXED STRATEGY

If a game does not have a saddle point, the two players cannot use maximin, minimax strategies (pure) as their optimal strategies, then the best strategies are mixed strategies. The two players, instead of selecting pure strategies only, may play their plays according to predetermined set which consists of probabilities corresponding to each of their pure strategies.

Table 3: Payoff matrix of two person zero sum game

(A's payoff matrix)							
Probabilities		y_1	y_2	…	y_j	…	y_n
	Pure Strategies	1	2	…	j	…	n
x_1	1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}
x_2	2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}
. . .	. . .	… … …	… … …	… … …	… … …	… … …	… … …
x_i	i	a_{i1}	a_{i2}	…	a_{ij}	…	a_{in}
. . .	. . .	… … …	… … …	… … …	… … …	… … …	… … …
x_m	m	a_{m1}	a_{m2}	…	a_{mj}	…	a_{mn}

Consider a rectangular game played by two players A (maximizing player) and B, with payoff matrix $[a_{ij}]_{m\times n}$. Here the players A and B have m and n pure strategies respectively.

Let $X = (x_1, x_2, x_3, \ldots\ldots, x_m)$ and $Y = (y_1, y_2, y_3, \ldots\ldots, y_n)$ be the mixed strategies of the two players A and B, respectively where $(x_1, x_2, x_3, \ldots\ldots, x_m)$ and $(y_1, y_2, y_3, \ldots\ldots, y_n)$ are the probabilities by which A and B, respectively, select their pure strategies, such that

$$\sum_{i=1}^{n} X_i = 1 \text{ And } \sum_{j=1}^{n} y_j = 1 \qquad \ldots\ldots\ldots (8)$$

And $x_i \geq 0, y_j \geq 0$ for all i=1,2,3….,m; j=1,2,3….,n.

Now expected gain to A is

$$a_{11}x_1 + a_{21}x_2 + \ldots. + a_{i1}x_i + \ldots. + a_{m1}x_m = \sum_{i=1}^{m} a_{i1}x_i \qquad \ldots\ldots\ldots (9)$$

(if B uses strategy 1 with probability y_1)

$$a_{12}x_1 + a_{22}x_2 + \ldots. + a_{i2}x_i + \ldots. + a_{m2}x_m = \sum_{i=1}^{m} a_{i2}x_i \qquad \ldots\ldots\ldots (10)$$

(if B uses strategy 2 with probability y_2)

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$$a_{j1}x_1 + a_{2j}x_2 + \ldots. + a_{ij}x_i + \ldots. + a_{mj}x_m = \sum_{i=1}^{m} a_{ij}x_i \qquad \ldots\ldots\ldots (11)$$

(if B uses strategy j with probability y_j)

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$$a_{1n}x_1 + a_{2n}x_2 + \ldots. + a_{in}x_n + \ldots. + a_{mn}x_m = \sum_{i=1}^{m} a_{in}x_i \qquad \ldots\ldots\ldots (12)$$

(if B uses strategy n with probability y_n)

The expected gain to A (payoff function to A) is given by

$$E(X,Y)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j \qquad \text{.......... (13)}$$

By maximin-minimax criterion, A selects $x_i(x_i \geq 0, \sum_{i=1}^{m} X_i = 1)$ which will maximize his minimum expected gain. i.e. A selects x_i which will

$$\max[\min\{\sum_{i=1}^{m}a_{i1}x_i, \sum_{i=1}^{m}a_{i2}x_i, ..., \sum_{i=1}^{m}a_{in}x_i\}] \qquad \text{.......... (14)}$$

This value is referred to as the maximin ($\underline{v}$) expected value for player A. Similarly B selects $y_j(y_j \geq 0, \sum_{j=1}^{n} y_j = 1)$ which will minimize his maximum expected loss, i.e. B selects y_j which will

$$\min[\max\{\sum_{j=1}^{n}a_{1j}y_j, \sum_{j=1}^{n}a_{2j}y_j, ..., \sum_{j=1}^{n}a_{mj}y_j\}] \qquad \text{.......... (15)}$$

This value is referred to as the minimax ($\bar{v}$) expected value for player B.

Hence, for A the best strategy is that which maximizes $\min\sum_{i=1}^{m}a_{ij}x_i$ and for B best strategy is that which minimizes $\max\sum_{j=1}^{n}a_{ij}y_j$. As in the case of pure strategies it can be shown that $\underline{v} \leq \bar{v}$, the fundamental theorem of rectangular games assumes that there always exists optimum strategies, such that $\bar{v} = \underline{v}$.

Lemma: This lemma asserts that when the row player plays x, among the most effective replies y of the column player, there is always at least one pure strategy. Symbolically,

$$\max_{y\in\Delta(s_2)} xAy = \min_{j}\sum_{i=1}^{m}a_{ij}x_i \qquad \text{.......... (16)}$$

Proof: For a given j, the summation

$$\sum_{i=1}^{m}a_{ij}x_i \qquad \text{.......... (17)}$$

Gives the payoff to the row player when she plays $x = (x_1, ..., x_m)$ and the column player the Pure strategy y_j. That is,

$$\sum_{i=1}^{m}a_{ij}x_i = u_i(x, y_j) \qquad \text{.......... (18)}$$

Therefore, $\min_{j}\sum_{i=1}^{m}a_{ij}x_i$ .... (19)

Gives the minimum payoff that the row player gets when plays x and when the column player plays only pure strategies. Since a pure strategy is a special case of mixed strategies, we have

$$\min_{j}\sum_{i=1}^{m}a_{ij}x_i \geq \min_{y\in\Delta(s_2)} xAy \qquad \text{.......... (20)}$$

On the other hand,

$$xAy = \sum_{j=1}^{n}y_j(\sum_{i=1}^{m}a_{ij}x_i) \qquad \text{.......... (21)}$$

$$r \geq \sum_{j=1}^{n}y_j(\min_{j}\sum_{i=1}^{m}a_{ij}x_i) \qquad \text{.......... (22)}$$

$$= \min_j \sum_{i=1}^{m} a_{ij} x_i \quad \text{Since} \quad \sum_{j=1}^{n} y_j = 1 \qquad \ldots\ldots\ldots. (23)$$

Therefore, we have:

$$xAy \geq \min_j \sum_{i=1}^{m} a_{ij} x_i \forall y \in \Delta(s_2); \forall x \in \Delta(s_1) \qquad \ldots\ldots\ldots. (24)$$

$$\min_{y \in \Delta(s_2)} xAy \geq \min_j \sum_{i=1}^{m} a_{ij} x_i \qquad \ldots\ldots\ldots. (25)$$

From (20) and (25), we have,

$$\min_{y \in \Delta(s_2)} xAy = \min_j \sum_{i=1}^{m} a_{ij} x_i \qquad \ldots\ldots\ldots. (26)$$

Similarly, it can be shown that

$$\max_{x \in \Delta(s_1)} xAy = \max_i \sum_{j=1}^{n} a_{ij} y_j \qquad \ldots\ldots\ldots. (27)$$

From the above lemma, we can describe the optimization problems of the row player and column players as follows.

Row Player's Optimization Problem (Max minimization)

$$\text{maximize} \min_j \sum_{i=1}^{m} a_{ij} x_i \qquad \ldots\ldots\ldots. (28)$$

Subject to

$$\sum_{i=1}^{m} x_i = 1 \forall x_i \geq 0; i = 1, 2, \ldots, m \qquad \ldots\ldots\ldots. (29)$$

Call the above problem P1. Note that this is equivalent to

$$\max_{x \in \Delta(s_1)} \min_{y \in \Delta(s_2)} xAy$$

Column Player's Optimization Problem (Min maximization)

$$\text{minimize} \max_i \sum_{j=1}^{n} a_{ij} y_j \qquad \ldots\ldots\ldots. (30)$$

Subject to

$$\sum_{j=1}^{n} y_i = 1 \forall y_j \geq 0; \text{j} = 1, 2, \ldots, \text{n} \qquad \ldots\ldots\ldots. (31)$$

Call the above problem P2. Note that this is equivalent to

$$\min_{y \in \Delta(s_2)} \max_{x \in \Delta(s_1)} xAy$$

We now show that the problems P1 and P2 are equivalent to appropriate linear programs.
Proposition: The following problems are equivalent.

$$\text{Maximize } \min_j \sum_{i=1}^{m} a_{ij} x_i \qquad \ldots\ldots\ldots. (32)$$

Subject to

$$\sum_{i=1}^{m} x_i = 1; P1 \qquad \ldots\ldots\ldots. (33)$$

$$x_i \geq 0; \text{i} = 1, 2, \ldots, \text{m}$$

Maximize z
Subject to

$$z - \min_{j} \sum_{i=1}^{m} a_{ij} x_i \leq 0;\ j = 1, 2, \ldots.., n \qquad \ldots\ldots\ldots. (34)$$

$$\sum_{i=1}^{m} x_i = 1; \text{LP1} \qquad \ldots\ldots\ldots. (35)$$

$$x_i \geq 0; \mathrm{i} = 1, 2, \ldots, \mathrm{m}$$

Proof: Note that P1 is a maximization problem and therefore by looking at the constraints

$$z - \sum_{i=1}^{m} a_{ij} x_i \leq 0;\ j = 1, 2, \ldots.., n \qquad \ldots\ldots\ldots. (36)$$

Any optimal solution z∗ will satisfy the equality in the above constraint. That is,

$$z^* = \sum_{i=1}^{m} a_{ij} x_i^* \ \text{ For some } \ j = 1, 2, \ldots.., n \qquad \ldots\ldots\ldots. (37)$$

Let j∗ be one such value of j. Then

$$z^* = \sum_{i=1}^{m} a_{ij} * x_i^* \qquad \ldots\ldots\ldots. (38)$$

Because z∗ is a feasible solution of LP1, we have

$$\sum_{i=1}^{m} a_{ij} * x_i^* \leq \sum_{i=1}^{m} a_{ij} x_i^* \forall j = 1, 2, \ldots., n \qquad \ldots\ldots\ldots. (39)$$

This means

$$\sum_{i=1}^{m} a_{ij} * x_i^* = \min_{j} \sum_{i=1}^{m} a_{ij} x_i^* \qquad \ldots\ldots\ldots. (40)$$

If not, we have

$$z^* < \sum_{i=1}^{m} a_{ij} x_i; \forall j = 1, 2, \ldots, n \qquad \ldots\ldots\ldots. (41)$$

If this happens, we can find a feasible solution $\hat{z}$ such that $\hat{z} > z^*$. Such a $\hat{z}$ is precisely the one for which equality will hold. But since z^* is a maximal value, the existence of $\hat{z} > z^*$ is a contradiction.
The summary so far is:
The row player's optimal strategy is max minimization:

$$\max_{x \in \Delta(s_1)} \max_{y \in \Delta(s_2)} xAy \qquad \ldots\ldots\ldots. (42)$$

This is equivalent to the following problem:

$$\text{Maximize } \min_{j} \sum_{i=1}^{m} a_{ij} x_i \qquad \ldots\ldots\ldots. (43)$$

Subject to P1

$$\sum x_i = 1; x_i \geq 0 \forall \mathrm{i} = 1, 2, \ldots, \mathrm{m} \qquad \ldots\ldots\ldots. (44)$$

The above is equivalent to the following LP:
Maximize z
Subject to

$$z - \sum_{i=1}^{m} a_{ij} x_i \leq 0;\ j = 1, 2, \ldots.., n \quad \text{LP1} \qquad \ldots\ldots\ldots. (45)$$

$$\sum x_i = 1; x_i \geq 0 \forall \mathrm{i} = 1, 2, \ldots, \mathrm{m} \qquad \ldots\ldots\ldots. (46)$$

The column player's optimal strategy is min maximization:

$$\min_{y \in \Delta(s_2)} \max_{x \in \Delta(s_1)} xAy \qquad \ldots\ldots\ldots. (47)$$

This is equivalent to:

$$\text{minimize} \max_i \sum_{j=1}^{n} a_{ij} y_j \qquad \text{……….} (48)$$

Subject to P2

$$\sum y_j = 1; y_j \geq 0 \forall \ \mathrm{j} = 1, 2, ..., \mathrm{n} \qquad \text{……….} (49)$$

The above is equivalent to the following LP:

Minimize w

Subject to

$$w - \sum_{j=1}^{n} a_{ij} x_i \geq 0; \mathrm{i} = 1, 2, ..., \mathrm{m} \quad \text{LP2} \qquad \text{……….} (50)$$

$$\sum y_j = 1; y_j \geq 0 \forall \ \mathrm{j} = 1, 2, ..., \mathrm{n} \qquad \text{……….} (51)$$

Minimax Theorem:

This result is one of the important landmarks in the initial decades of game theory. This result was proved by Von Neumann in 1928 using the Brower's fixed point theorem. Later, he and Morgenstern provided an elegant proof of this theorem using LP duality. The key implication of the minimax theorem is the existence of a mixed strategy Nash equilibrium in any matrix game.

***Theorem*:** For every (m × n) matrix A, there is a stochastic row vector $x^* = (\mathrm{x}_1^*, ..., \mathrm{x}_m^*)$ and a stochastic column vector $y^* = (\mathrm{y}_1^*, ..., \mathrm{y}_n^*)^T$ such that

$$\min_{y \in \Delta(\mathrm{s}_2)} x^* A y = \max_{x \in \Delta(\mathrm{s}_1)} x A y^* \qquad \text{……….} (52)$$

Proof: Given a matrix A, we have derived linear programs LP1, LP2 where LP1 represents the optimal Strategy of row player while LP2 represents the optimal strategy of column player. First, it is observed that the linear program LP2 is the dual of the linear program LP1. We now invoke the strong duality theorem which says: If an LP has an optimal solution, then its dual also has an Optimal solution; moreover the optimal value of the dual is the same as the optimal value of the original (primal) LP.

To apply the strong duality theorem in the current context, we first observe that the problem P1 has an optimal solution by the very nature of the problem. Since LP1 is equivalent to the problem P1, the immediate implication is that LP1 has an optimal solution. Thus we have two LPs, LP1 and LP2 which are duals of each other and LP1 has an optimal solution, then by the strong duality theorem, LP2 also has an optimal solution and the optimal value of LP2 is the same as the optimal value of LP1.

Let $z^*, \mathrm{x}_1^*, ..., \mathrm{x}_m^*$ be an optimal solution of LP1. Then, we have

$$z^* = \sum_{i=1}^{m} a_{ij} * x_i^* \ \text{ For some } \ j^* \in 1, 2, ..., n \qquad \text{……….} (53)$$

By the feasibility of the optimal solution in LP1, we have

$$\sum_{i=1}^{m} a_{ij} * x_i^* \leq \sum_{i=1}^{m} a_{ij} x_i^* \forall j = 1, 2, ..., n \qquad \text{……….} (54)$$

This implies that

$$\sum_{i=1}^{m} a_{ij} * x_i^* = \min_j \sum_{i=1}^{m} a_{ij} x_i^* \qquad \text{……….} (55)$$

$$= \min_{y \in \Delta(\mathrm{s}_2)} x^* A y \quad \text{(By the lemma)} \qquad \text{……….} (56)$$

Thus

$$z^* = \min_{y \in \Delta(\mathrm{s}_2)} x^* A y \qquad \text{……….} (57)$$

Similarly, let $w^*, y_1^*, ..., y_n^*$ be an optimal solution of LP2. Then

$$w^* = \sum_{j=1}^{n} a_{i^*j} y_j^* \quad \text{For some} \quad j^* \in 1, 2, ..., m \qquad .... (58)$$

By the feasibility of the optimal solution in LP2, we have

$$\sum_{j=1}^{m} a_{i^*j} y_j^* \geq \sum_{i=1}^{n} a_{ij} y_i^* \forall j = 1, 2, ..., m \qquad .... (59)$$

$$\sum_{j=1}^{n} a_{i^*j} y_j^* = \max_i \sum_{j=1}^{n} a_{ij} y_j^* \qquad .... (60)$$

$$= \max_{x \in \Delta(s_1)} xAy^* \quad \text{(By Lemma)} \qquad .... (61)$$

Therefore

$$w^* = \max_{x \in \Delta(s_1)} xAy^* \qquad .... (62)$$

By the strong duality theorem, the optimal values of the primal and the dual are the same and Therefore $z^* = w^*$ This means

$$\min_{y \in \Delta(s_2)} x^* Ay = \max_{x \in \Delta(s_1)} xAy^* \qquad .... (63)$$

This proves the minimax theorem.

We now show that the mixed strategy profile (x^*, y^*) is in fact a mixed strategy Nash equilibrium of the matrix game A. For this, consider

$$x^* Ay^* \geq \min_{y \in \Delta(s_2)} x^* Ay \qquad .... (64)$$

$$= \max_{x \in \Delta(s_1)} xAy^* \qquad .... (65)$$

$$\geq xAy^* \forall x \in \Delta(s_1) \qquad .... (66)$$

That is,

$$x^* Ay^* \geq xAy^* \forall x \in \Delta(s_1) \qquad .... (67)$$

This implies

$$u_1(x^*, y^*) \geq u_1(x, y^*) \forall x \in \Delta(s_1) \qquad .... (68)$$

Further

$$x^* Ay^* \leq \max_{x \in \Delta(s_1)} xAy^* \qquad .... (69)$$

$$= \min_{x \in \Delta(s_2)} x^* Ay \qquad .... (70)$$

$$\geq x^* Ay \forall y \in \Delta(s_2) \qquad .... (71)$$

That is,

$$x^* Ay^* \leq x^* Ay \forall y \in \Delta(s_2) \qquad .... (72)$$

This implies

$$u_2(x^*, y^*) \geq u_2(x^*, y) \forall y \in \Delta(s_2) \qquad .... (73)$$

Thus (x^*, y^*) is a mixed strategy Nash equilibrium or a randomized saddle point. This means the minimax theorem guarantees the existence of a mixed strategy Nash equilibrium for any matrix game.

A KEY THEOREM FOR NASH EQUILIBRIUM

We now state and prove a key theorem for a mixed strategy profile to be Nash equilibrium in matrix games.

Theorem: Given a two player zero sum game

$(\{1, 2\}, s_1, s_2, u_1, -u_1)$

a mixed strategy profile (x∗, y∗) is a Nash equilibrium if and only if

$$x^{*} \in \arg\max_{x\in\Delta(s_1)} \min_{y\in\Delta(s_2)} xAy \qquad \text{..........} (74)$$

And

$$y^{*} \in \arg\min_{y\in\Delta(s_2)} \max_{x\in\Delta(s_1)} xAy \qquad \text{..........} (75)$$

Furthermore

$$u_1(x^*, y^*) = -u_2(x^*, y^*) \qquad \text{..........} (76)$$

$$\max_{x\in\Delta(s_1)} \min_{y\in\Delta(s_2)} xAy \qquad \text{..........} (77)$$

$$\min_{y\in\Delta(s_2)} \max_{x\in\Delta(s_1)} xAy \qquad \text{..........} (78)$$

***Proof*:** Suppose (x∗, y∗) is Nash equilibrium. Then

$$u_1(x^*, y^*) \ge u_1(x, y^*) \forall x \in \Delta(s_1)$$

$$\Rightarrow u_1(x^*, y^*) = \max_{x\in\Delta(s_1)} u_1(x, y^*) \qquad \text{..........} (79)$$

Also, note that

$$u_1(x, y^*) \ge \min_{y\in\Delta(s_2)} u_1(x, y) \forall \, x \in \Delta(s_1)$$

$$\Rightarrow \max_{x\in\Delta(s_1)} u_1(x, y^*) \ge \max_{x\in\Delta(s_1)} \{ \min_{y\in\Delta(s_2)} u_1(x, y)\} \qquad \text{..........} (80)$$

Since

$$f(x) \ge g(x) \forall x \Rightarrow \max_x f(x) \ge \max_x g(x) \qquad \text{..........} (81)$$

From (79) and (80), we get

$$u_1(x^*, y^*) \ge \max_{x\in\Delta(s_1)} \{ \min_{y\in\Delta(s_2)} u_1(x, y)\} \qquad \text{..........} (82)$$

On similar lines, using, $u_1(x^*, y^*) = -u_2(x^*, y^*)$ we can show that

$$u_1(x^*, y^*) \le \min_{y\in\Delta(s_2)} \{ \max_{x\in\Delta(s_1)} u_1(x, y)\} \qquad \text{..........} (83)$$

We have, $u_1(x^*, y^*) = -u_2(x^*, y^*)$ .... (84)

$$= -\{ \max_{y\in\Delta(s_2)} u_2(x^*, y)\} \qquad \text{..........} (85)$$

$$= \min_{y\in\Delta(s_2)} \{u_2(x^*, y)\}$$

$$= \min_{y\in\Delta(s_2)} u_1(x^*, y) \qquad \text{..........} (86)$$

$$u_1(x^*, y^*) = \min_{y\in\Delta(s_2)} u_1(x^*, y) \qquad \text{..........} (87)$$

We know that

$$\max_{x\in\Delta(s_1)} \min_{y\in\Delta(s_2)} u_1(x, y) \ge \min_{y\in\Delta(s_2)} u_1(x^*, y) \qquad \text{..........} (88)$$

$$= u_1(x^*, y^*) \quad \text{By (82)} \qquad \text{..........} (89)$$

Similarly we know that

$$\min_{y\in\Delta(s_2)} \max_{x\in\Delta(s_1)} u_1(x, y) \ge \max_{x\in\Delta(s_1)} u_1(x, y^*) \qquad \text{..........} (90)$$

$$= u_1(x^*, y^*) \qquad \text{..........} (91)$$

Equation (79) and (83) imply that

$$u_1(x^*, y^*) = \max_{x\in\Delta(s_1)} \min_{y\in\Delta(s_2)} u_1(x, y) \qquad \text{..........} (92)$$

Equation (80) and (87) imply that

$$u_1(x^*, y^*) = \min_{y\in\Delta(s_2)} \max_{x\in\Delta(s_1)} u_1(x, y) \qquad \text{..........} (93)$$

From the above two expressions, we have

$$x^* \in \arg\max_{x \in \Delta(s_1)} \min_{y \in \Delta(s_2)} u_1(x, y) \qquad \text{.......... (94)}$$

$$y^* \in \arg\min_{y \in \Delta(s_2)} \max_{x \in \Delta(s_1)} u_1(x, y) \qquad \text{.......... (95)}$$

It completes the proof. It shows that (x∗, y∗) is a Nash equilibrium.
A flow chart of using game theory approach to solve the problem is shown by the following figure-2.

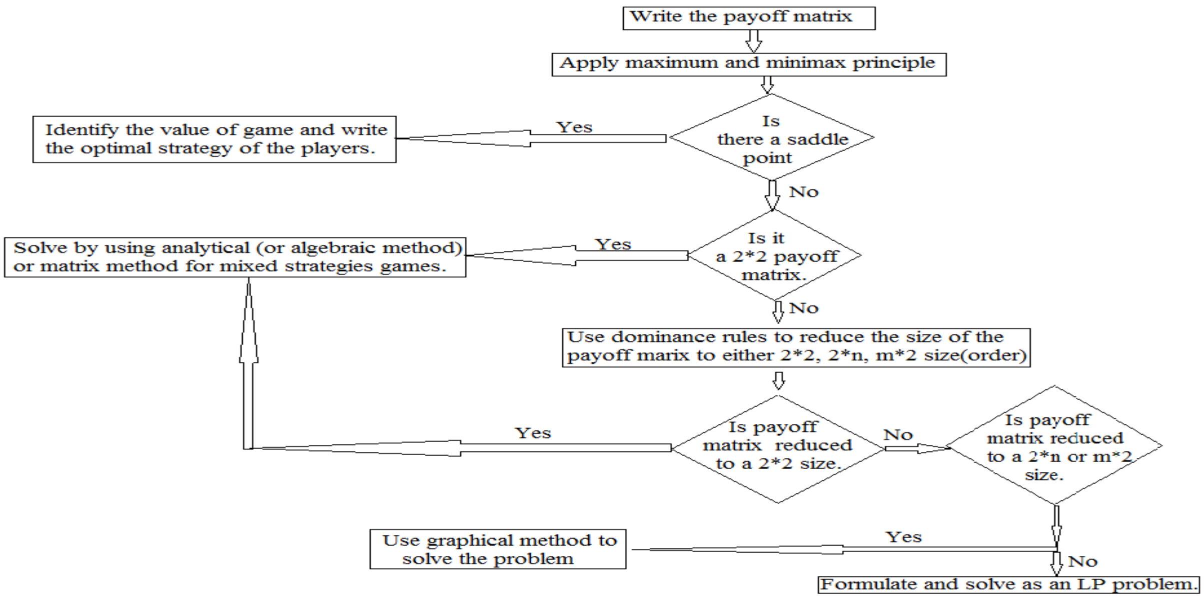

Figure-2: Flow chart

Method of solution of 2×2 games without saddle point:

Here we consider the 2×2 game which does not have a saddle point. So in this case the best strategies are the mixed strategies i.e. here we shall determine the probabilities with which each strategy should be selected. Suppose the game is

	Player B	
Player A	p_{11}	p_{12}
	p_{21}	p_{22}

Table 4: 2×2 game without saddle point

We assume the probability that player A commences with the 1st row to be 'm' and that for commencing with 2nd row to be 1-m.

Also, we assume the probability that player B commences with the 1st row to be 'n' and that for commencing with 2nd row to be 1-n.

If B plays the 1st column, then A gains

$$p_{11}m + p_{21}(1-m) \qquad \text{.......... (96)}$$

If B plays the 2nd column, then A gains

$$p_{12}m + p_{22}(1-m) \qquad \text{.......... (97)}$$

Taking result (1) & (2) as identical, we get

$$p_{11}m + p_{21}(1-m) = p_{12}m + p_{22}(1-m) \qquad \text{.......... (98)}$$

Solving and simplifying, we get

$$m = \frac{(p_{22} - p_{21})}{(p_{11} - p_{21}) + (p_{22} - p_{12})} \qquad \text{………. (99)}$$

and

$$1 - m = \frac{(p_{11} - p_{12})}{(p_{11} - p_{12}) + (p_{22} - p_{21})} \qquad \text{……… (100)}$$

Proceeding in a similar manner, we can find the optimal strategies for B. if A plays the 1st row then the gain to B would be

$$p_{11}n + p_{12}(1-n) \qquad \text{……… (101)}$$

If A play the 2nd roe, then gain to B would be

$$p_{21}n + p_{22}(1-n) \qquad \text{………. (102)}$$

From results (101) and (102), we get

$$p_{11}n + p_{12}(1-n) = p_{21}n + p_{22}(1-n) \qquad \text{……….. (103)}$$

Solving and simplifying

$$n = \frac{(p_{22} - p_{12})}{(p_{11} - p_{21}) + (p_{22} - p_{12})} \qquad \text{………. (104)}$$

and

$$1 - n = \frac{(p_{11} - p_{21})}{(p_{11} - p_{21}) + (p_{22} - p_{12})} \qquad \text{………. (105)}$$

Now, if A plays (m, 1-m) then value of the game is

$$= p_{11}m + p_{21}(1-m) \qquad \text{………. (106)}$$

Substituting the value of ‘m’ & ‘1-m’ in the above result, we get the value as

$$= \frac{p_{11}p_{22} - p_{11}p_{21} + p_{21}p_{11} - p_{21}p_{12}}{(p_{11} - p_{12}) + (p_{22} - p_{21})} \qquad \text{……… (107)}$$

NUMERICAL EXAMPLES

Example-1: Solve the following game

	Player B		
		I	II
Player A	I	8	2
	II	6	10

Solution:

Method-I

It may be noted that the game has no saddle point and thus the best strategy for both players are mixed strategies. Let player A’s strategy be (m, 1-m) and player B be (n, 1-n). Then we know that the values of m, n for algebraic method are follows:

$$\mathrm{m}=\frac{(p_{22}-p_{21})}{(p_{11}-p_{21})+(\mathrm{p}_{22}-p_{12})}=\frac{10-6}{(8-6)+(10-2)}=\frac{2}{5}$$

$$1-m=1-\frac{2}{5}=\frac{3}{5}$$

$$n=\frac{(p_{22}-p_{12})}{(p_{11}-p_{21})+(\mathrm{p}_{22}-p_{12})}=\frac{10-2}{(8-6)+(10-2)}=\frac{4}{5}$$

$$1-n=1-\frac{4}{5}=\frac{1}{5}$$

The value of game by algebraic method is

$$v=\frac{p_{11}p_{22}-p_{12}p_{21}}{p_{11}+p_{22}-(p_{12}+p_{21})}=\frac{8\times 10-6\times 2}{8+10-(2+6)}=\frac{68}{10}=\frac{34}{5} \qquad \text{.......... (108)}$$

Method-II

We have the value of m and n from above method that is

$$m=\frac{2}{5},\mathrm{n}=\frac{4}{5},1-m=\frac{3}{5} and 1-n=\frac{1}{5}$$

Now the value of game by calculus method is

$$E(\mathrm{m,n})=p_{11}mn+p_{21}(1-m)\,\mathrm{n}+p_{12}m(1-n)+p_{22}(1-m)(1-n)$$

$$E(\mathrm{m,n})=8\times\frac{2}{5}\times\frac{4}{5}+6\times\frac{3}{5}\times\frac{4}{5}+2\times\frac{2}{5}\times\frac{1}{5}+10\times\frac{3}{5}\times\frac{1}{5}$$

$$E(\mathrm{m,n})=\frac{64+72+4+30}{25}=\frac{170}{25}=\frac{34}{5} \qquad \text{.......... (109)}$$

Method-III

Find the value of game by new approach

The probabilities can be find out by either algebraic or calculus method and then employed to find the value of game.

The value of m, 1-m, n, 1-n is

$$m=\frac{2}{5},\mathrm{n}=\frac{4}{5},1-m=\frac{3}{5} and 1-n=\frac{1}{5}$$

Then the two probabilities for player A and B are $(\frac{2}{5},\frac{3}{5})$ and $(\frac{4}{5},\frac{1}{5})$.

Since both players play independently and that neither knows that the other will play next. The probabilities for player A are independent of the probabilities for player B.

The payoff in the game will be obtained when players play a particular column and particular row simultaneously. Once the probabilities for choosing a particular row and column are known. The probabilities of each payoff can be calculated and thereby expected value can be found (expected value= payoff × probability of payoff) as follow:

Table 5: Payoff matrix

Payoff value	Strategy	Probability of the payoff	Expected value
8	Row-I, Column-I	$\frac{2}{5}\times\frac{4}{5}=\frac{8}{25}$	$\frac{64}{25}$
2	Row-I, Column-II	$\frac{2}{5}\times\frac{1}{5}=\frac{2}{25}$	$\frac{4}{25}$
6	Row-II, Column-I	$\frac{3}{5}\times\frac{4}{5}=\frac{12}{25}$	$\frac{72}{25}$
10	Row-II, Column-II	$\frac{3}{5}\times\frac{1}{5}=\frac{3}{25}$	$\frac{30}{25}$
	Total value	$\frac{25}{25}=1$	$\frac{34}{5}$

Thus the value of game is $\frac{34}{5}$.

Which is similar as algebraic and calculus method.

Example-2: Solve the following game

	Player B		
Player A		I	II
	I	6	9
	II	8	4

Solution:

Method-I

It may be noted that the game has no saddle point and thus the best strategy for both players are mixed strategies. Let player A's strategy be (m, 1-m) and player B be (n, 1-n). Then we know that the values of m, n for algebraic method are follows:

$$\mathrm{m}=\frac{(p_{22}-p_{21})}{(p_{11}-p_{21})+(\mathrm{p}_{22}-p_{12})}=\frac{4-8}{(6-8)+(4-9)}=\frac{4}{7}$$

$$1-m=1-\frac{4}{7}=\frac{3}{7}$$

$$n=\frac{(p_{22}-p_{12})}{(p_{11}-p_{21})+(\mathrm{p}_{22}-p_{12})}=\frac{4-9}{(6-8)+(4-9)}=\frac{5}{7}$$

$$1-n=1-\frac{5}{7}=\frac{2}{7}$$

The value of game by algebraic method is

$$v=\frac{p_{11}p_{22}-p_{12}p_{21}}{p_{11}+p_{22}-(p_{12}+p_{21})}=\frac{6\times4-9\times8}{6+4-(9+8)}=\frac{24-72}{10-17}=\frac{48}{7} \qquad \text{……….} (110)$$

Method-II

We have the value of m and n from above method that is

$$m=\frac{4}{7},\mathrm{n}=\frac{5}{7},1-m=\frac{3}{7} and 1-n=\frac{2}{7}$$

Now the value of game by calculus method is

$$E(\mathrm{m},\mathrm{n})=p_{11}mn+p_{21}(1-m)\,\mathrm{n}+p_{12}m(1-n)+p_{22}(1-m)(1-n)$$

$$E(\mathrm{m},\mathrm{n})=6\times\frac{4}{7}\times\frac{5}{7}+8\times\frac{3}{7}\times\frac{5}{7}+9\times\frac{4}{7}\times\frac{2}{7}+4\times\frac{3}{7}\times\frac{2}{7}$$

$$E(\mathrm{m},\mathrm{n})=\frac{120+120+72+24}{7\times7}=\frac{336}{49}=\frac{48}{7} \qquad \text{………. (111)}$$

Method-III

Find the value of game by new approach

The probabilities can be find out by either algebraic or calculus method and then employed to find the value of game.

The value of m, 1-m, n, 1-n is

$$m=\frac{4}{7},\mathrm{n}=\frac{5}{7},1-m=\frac{3}{7} and 1-n=\frac{2}{7}$$

Then the two probabilities for player A and B are $(\frac{4}{7},\frac{3}{7})$ and $(\frac{5}{7},\frac{2}{7})$.

Since both players play independently and that neither knows that the other will play next. The probabilities for player A are independent of the probabilities for player B.

The payoff in the game will be obtained when players play a particular column and particular row simultaneously. Once the probabilities for choosing a particular row and column are known. The probabilities of each payoff can be calculated and thereby expected value can be found (expected value= payoff × probability of payoff) as follow:

Table 6: Payoff matrix

Payoff value	Strategy	Probability of the payoff	Expected value
6	Row-I, Column-I	$\frac{4}{7}\times\frac{5}{7}=\frac{20}{49}$	$\frac{120}{49}$
9	Row-I, Column-II	$\frac{4}{7}\times\frac{2}{7}=\frac{8}{49}$	$\frac{72}{49}$
8	Row-II, Column-I	$\frac{3}{7}\times\frac{5}{7}=\frac{15}{49}$	$\frac{120}{49}$
4	Row-II, Column-II	$\frac{3}{7}\times\frac{2}{7}=\frac{6}{49}$	$\frac{24}{49}$
	Total value	$\frac{49}{49}=1$	$\frac{336}{49}=\frac{48}{7}$

Thus the value of game is $\frac{48}{7}$.

Which is similar as algebraic and calculus method.

Example-3: Solve the game whose payoff matrix is given below:

	Player B				
		I	II	III	IV
Player A	A	3	2	4	0
	B	3	4	2	4
	C	4	2	4	0
	D	0	4	0	8

Solution:

It is clear that this game has no saddle point. Therefore, we try to reduce the size of the given payoff matrix by dominance principles.

From player A point of view, the first row is dominated by the third row, yielding the reduced 3×4 payoff matrix. In the reduced matrix from player B point of view, the first column is dominated by the third column. Thus, by deleting the first row and then the first column, the reduced payoff matrix so obtained is:

Table 7: Reduced payoff matrix

	Player B			
		II	III	IV
Player A	B	4	2	4
	C	2	4	0
	D	4	0	8

Now it may be noted that none of the pure strategies of player A and player B is inferior to any of their other strategies. However, the average of payoffs due to strategies III and IV, [(2+4)/2; (4+0)/2; (0+8)/2] = (3, 2, 4) is superior to the payoff due to strategy II of player B. thus, strategy II may be deleted from the matrix. The new matrix so obtained is

Table 8: Payoff matrix

	Player B		
		III	IV
Player A	B	2	4
	C	4	0
	D	0	8

Again in the reduced matrix, the average of the payoffs due to strategies C and D of player A, i.e. [(4+0)/2;(0+8)/2]=(2,4) is the same as the payoff due to strategy Therefore, player A will gain the same amount even if the strategy B is never used. Hence, after deleting the strategy B from the reduced matrix the following new reduced 2×2 payoff is obtained

Table 9: Payoff matrix

	Player B		
Player A		III	IV
	C	4	0
	D	0	8

This game has no saddle point, so we solved this payoff matrix by algebraic, calculus and new approach as follows:

Method-I

It may be noted that the game has no saddle point and thus the best strategy for both players are mixed strategies. Let player A's strategy be (m, 1-m) and player B be (n, 1-n). Then we know that the values of m, n for algebraic method are follows:

$$\mathrm{m} = \frac{(p_{22} - p_{21})}{(p_{11} - p_{21}) + (\mathrm{p}_{22} - p_{12})} = \frac{8-0}{(4-0)+(8-0)} = \frac{8}{12} = \frac{2}{3}$$

$$1 - m = 1 - \frac{2}{3} = \frac{1}{3}$$

$$n = \frac{(p_{22} - p_{12})}{(p_{11} - p_{21}) + (\mathrm{p}_{22} - p_{12})} = \frac{8-0}{(4-0)+(8-0)} = \frac{8}{12} = \frac{2}{3}$$

$$1 - n = 1 - \frac{2}{3} = \frac{1}{3}$$

The value of game by algebraic method is

$$v = \frac{p_{11}p_{22} - p_{12}p_{21}}{p_{11} + p_{22} - (p_{12} + p_{21})} = \frac{4\times 8 - 0\times 0}{4+8-(0+0)} = \frac{32-0}{12-0} = \frac{32}{12} = \frac{8}{3} \quad \text{.......... (112)}$$

Method-II

We have the value of m and n from above method that is

$$m = \frac{2}{3}, \mathrm{n} = \frac{2}{3}, 1-m = \frac{1}{3} \, and \, 1-n = \frac{1}{3}$$

Now the value of game by calculus method is

$$E(\mathrm{m,n}) = p_{11}mn + p_{21}(1-m)\mathrm{n} + p_{12}m(1-n) + p_{22}(1-m)(1-n)$$

$$E(\mathrm{m,n}) = 4\times\frac{2}{3}\times\frac{2}{3} + 0\times\frac{1}{3}\times\frac{2}{3} + 0\times\frac{2}{3}\times\frac{1}{3} + 8\times\frac{1}{3}\times\frac{1}{3}$$

$$E(\mathrm{m,n}) = \frac{16+0+0+8}{9} = \frac{24}{9} = \frac{8}{3} \quad \text{.......... (113)}$$

Method-III

Find the value of game by new approach

The probabilities can be find out by either algebraic or calculus method and then employed to find the value of game.

The value of m, 1-m, n, 1-n is

$m=\frac{2}{3}, \mathrm{n}=\frac{2}{3}, 1-m=\frac{1}{3}$ and $1-n=\frac{1}{3}$.

Then the two probabilities for player A and B are. $(\frac{2}{3},\frac{1}{3})$ and $(\frac{2}{3},\frac{1}{3})$.

Since both players play independently and that neither knows that the other will play next. The probabilities for player A are independent of the probabilities for player B.

The payoff in the game will be obtained when players play a particular column and particular row simultaneously. Once the probabilities for choosing a particular row and column are known. The probabilities of each payoff can be calculated and thereby expected value can be found (expected value= payoff × probability of payoff) as follow:

Table 10: Payoff matrix

Payoff value	Strategy	Probability of the payoff	Expected value
4	Row-I, Column-I	$\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$	$\frac{16}{9}$
0	Row-I, Column-II	$\frac{2}{3}\times\frac{1}{3}=\frac{2}{9}$	0
0	Row-II, Column-I	$\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$	0
8	Row-II, Column-II	$\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$	$\frac{8}{9}$
	Total value	$\frac{9}{9}=1$	$\frac{24}{9}=\frac{8}{3}$

Thus the value of game is $\frac{8}{3}$.

Which is similar as algebraic and calculus method.

CONCLUSION

In this paper, we used a new approach to solve Two Person Zero Sum game without saddle point. Comparing the result for two person zero sum game without saddle point with algebraic and calculus method. These methods are already defined. It has been observed the value of the game is same for all the methods. This method is also applicable for higher order game.

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