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Wormhole with Isotropic Coordinates and a Possible Solution

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ABSTRACT

In this paper we consider the case of wormhole metric in terms of isotropic coordinates which considered by Matt Visser in his Lorentzian wormholes: From Einstein to Hawking and also study traversability and matter condition at the throat and at last consider about a possible solution of it.

KEYWORDS: Wormhole, Traversability Condition, Wormhole Solution.

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INTRODUCTION.

Wormhole solutions of Einstein equations, from the topological point of view, can be seen as to connecting two asymptotically flat regions of the space-time manifold, i.e. a short-cut or a bridge linking together two distant regions of the same space-time. Such solutions were conceived as particle models by Einstein himself in 1935 (Einstein-Rosen bridge[1,2]). However, an Einstein-Rosen solution turns out to be just a portion of the Schwarzschild's metric[3,4] describing a Black Hole, therefore the bridge cannot be crossed[5,6]. In 1988, the pioneer work by K. Thorne and M. Morris[7,8] led to a more deep understanding of wormhole physics, in particular introducing the concept of a traversable wormhole. The authors in Ref.[7] showed that in order to achieve traversability one has to demand the metric to be event horizons free. This feature, combined with Birkhoff's theorem, implies that a traversable asymptotically flat wormhole must be a solution of Einstein equation in presence of some kind of matter (so-called "exotic matter") with non standard stress-energy tensor, namely in a neighborhood of the wormhole throat the radial tension must exceed the total energy density . The region[7] around the throat, where such matter is confined , shows a violation of the Null Energy Condition (NEC).

The paper is organized as follows. In Section II we briefly describes Einstein field equations and a traversable wormhole with space-time metric in terms of isotropic coordinates and from the flare out condition we see the violation of the Null Energy Condition (NEC). Here in section III, also considering the traversability condition and give a shape function as wormhole solution for zero tidal force.

MATHEMATICAL DETAILS OF THE WORMHOLE WITH ISTROPIC COORDINATES.

Form of the Metric.

In this case space-time metric be written as

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{-2\psi(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] \tag{1}$$

where the quantities $\psi(r)$ and $\phi(r)$denote the so-called shape and red-shift functions.

This geometry describes a traversable wormhole provided

(i) $\psi(r)$ and $\phi(r)$ are everywhere finite.

(ii) $C(r) \equiv 2\pi r\, e^{-2\psi(r)}$ has a minimum at $r_0 \neq 0$. Where r_0 is the location of throat.

(iii) $\phi(0), \phi(\infty)$ and $\psi(\infty)$ must be finite.

Equations of Structure for the Wormhole.

From the metric expressed in the form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, one may determine the Christoffel symbols, defined as

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\, g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}).$$

Here the non-zero components are

$$\Gamma^t{}_{rt} = \phi', \quad \Gamma^r{}_{tt} = e^{2\psi(r)}e^{2\phi(r)}\,\phi'\,, \; \Gamma^r{}_{rr} = -\psi', \quad \Gamma^r{}_{\theta\theta} = -r(1-r\psi')\,,$$

$$\Gamma^r{}_{\varphi\varphi} = -r(1-r\psi')\sin^2\theta, \quad \Gamma^\theta{}_{r\theta} = \frac{1}{r}\,(1-r\psi')\,, \quad \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta\,, \tag{2}$$

$$\Gamma^\varphi{}_{r\varphi} = \frac{1}{r}\,(1-r\psi')\,, \quad \Gamma^\varphi{}_{\theta\varphi} = \cot\theta,$$

The Riemann tensor is defined as

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\lambda\gamma}\,\Gamma^\lambda{}_{\beta\delta} - \Gamma^\alpha{}_{\lambda\delta}\,\Gamma^\lambda{}_{\beta\gamma}$$

Applying this equation one can get

$$R^t{}_{rtr} = -\phi'' - \phi'(\psi'+\phi'), R^t{}_{\theta t\theta} = -\,\phi' r(1-r\psi')\,, R^t{}_{\varphi t\varphi} = -\,\phi' r(1-r\psi')\sin^2\theta,$$

$$R^r{}_{\theta r\theta} = r^2(\psi'' + \frac{\psi'}{r})\,, R^r{}_{\varphi r\varphi} = r^2(\psi'' + \frac{\psi'}{r})\sin^2\theta\,, R^\theta{}_{\varphi\theta\varphi} = r^2\psi'(\frac{2}{r} - \psi')\sin^2\theta \tag{3}$$

Using the orthonormal basis given by

$$\begin{cases} \boldsymbol{e}_{\hat{t}} = e^{-\phi}\boldsymbol{e}_t \\ \boldsymbol{e}_{\hat{r}} = e^{\psi}\boldsymbol{e}_r \\ \boldsymbol{e}_{\hat{\theta}} = e^{\psi}r^{-1}\boldsymbol{e}_\theta \\ \boldsymbol{e}_{\hat{\varphi}} = e^{\psi}(r\sin\theta)^{-1}\boldsymbol{e}_\varphi \end{cases} \tag{4}$$

the matrix of the metric coefficient is given by

$$g_{\hat{\alpha}\hat{\beta}} = \boldsymbol{e}_{\hat{\alpha}}.\boldsymbol{e}_{\hat{\beta}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Again multilinear object given by

$$R^t{}_{rtr}\,\boldsymbol{e}_t \otimes \boldsymbol{\omega}^r \otimes \boldsymbol{\omega}^t \otimes \boldsymbol{\omega}^r = e^{2\psi}R^t{}_{rtr}\boldsymbol{e}_{\hat{t}} \otimes \boldsymbol{\omega}^{\hat{r}} \otimes \boldsymbol{\omega}^{\hat{t}} \otimes \boldsymbol{\omega}^{\hat{r}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}\,\boldsymbol{e}_{\hat{t}} \otimes \boldsymbol{\omega}^{\hat{r}} \otimes \boldsymbol{\omega}^{\hat{t}} \otimes \boldsymbol{\omega}^{\hat{r}}$$

$$\Rightarrow R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = e^{2\psi}R^t{}_{rtr} = -e^{2\psi}[\phi'' + \phi'(\psi'+\phi')]$$

Where $\{\boldsymbol{\omega}^{\hat{t}}, \boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\varphi}}\}$ is the dual basis of $\{\boldsymbol{e}_{\hat{t}}\,, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}\,, \boldsymbol{e}_{\hat{\varphi}}\}$ and

$$\begin{cases} \boldsymbol{\omega}^t = e^{-\phi}\boldsymbol{\omega}^{\hat{t}} \\ \boldsymbol{\omega}^r = e^{\psi}\boldsymbol{\omega}^{\hat{r}} \\ \boldsymbol{\omega}^\theta = e^{\psi}r^{-1}\boldsymbol{\omega}^{\hat{\theta}} \\ \boldsymbol{\omega}^\varphi = e^{\psi}(r\sin\theta)^{-1}\boldsymbol{\omega}^{\hat{\varphi}} \end{cases} \tag{5}$$

Proceeding similarly, we find the 24 nonzero components of the Riemann tensor given by

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{r}\hat{t}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{t}\hat{r}} = -e^{2\psi}[\phi'' + \phi'(\psi'+\phi')]$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -R^{\hat{t}}{}_{\hat{\theta}\hat{\theta}\hat{t}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} = R^{\hat{\theta}}{}_{\hat{t}\hat{t}\hat{\theta}} = -e^{2\psi}(\tfrac{1}{r} - \psi')\phi'$$

$$R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} = -R^{\hat{t}}{}_{\hat{\varphi}\hat{\varphi}\hat{t}} = -R^{\hat{\varphi}}{}_{\hat{t}\hat{\varphi}\hat{t}} = R^{\hat{\varphi}}{}_{\hat{t}\hat{t}\hat{\varphi}} = -e^{2\psi}(\tfrac{1}{r} - \psi')\phi' \tag{6}$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = -R^{\hat{r}}{}_{\hat{\theta}\hat{\theta}\hat{r}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = e^{2\psi}(\psi'' + \tfrac{\psi'}{r})$$

$$R^{\hat{r}}{}_{\hat{\varphi}\hat{r}\hat{\varphi}} = -R^{\hat{r}}{}_{\hat{\varphi}\hat{\varphi}\hat{r}} = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}} = -R^{\hat{\phi}}{}_{\hat{r}\hat{r}\hat{\varphi}} = e^{2\psi}(\psi'' + \tfrac{\psi'}{r})$$

$$R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = -R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\varphi}\hat{\theta}} = R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\theta}\hat{\varphi}} = e^{2\psi}\psi'(\tfrac{2}{r} - \psi')$$

From where we can contract the Riemann tensor to calculate the Ricci tensor $R_{\hat{\alpha}\hat{\beta}}$ and the scalar curvature R, given by

$R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}\hat{\nu}}$ and $R = g^{\hat{\mu}\hat{\nu}}R_{\hat{\mu}\hat{\nu}} = -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}} + R_{\hat{\varphi}\hat{\varphi}}$ and from this we can compute Einstein tensor given by $G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}Rg_{\hat{\mu}\hat{\nu}}$. Therefore the only nonzero components of Einstein tensor are

$$G_{\hat{t}\hat{t}} = R_{\hat{t}\hat{t}} - \tfrac{1}{2}Rg_{\hat{t}\hat{t}} = R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\varphi}\hat{r}\hat{\varphi}} + R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = e^{2\psi}(2\psi'' - (\psi')^2 + \tfrac{4\psi'}{r}) \tag{7}$$

$$G_{\hat{r}\hat{r}} = R_{\hat{r}\hat{r}} - \tfrac{1}{2}Rg_{\hat{r}\hat{r}} = -R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} - R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} - R^{\hat{\theta}}{}_{\hat{\varphi}\hat{\theta}\hat{\varphi}} = e^{2\psi}\left[(\psi')^2 - 2\phi'\psi' + \tfrac{2(\phi' - \psi')}{r}\right] \tag{8}$$

$$G_{\hat{\varphi}\hat{\varphi}} = G_{\hat{\theta}\hat{\theta}} = R_{\hat{\theta}\hat{\theta}} - \tfrac{1}{2}Rg_{\hat{\theta}\hat{\theta}} = -R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} - R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} - R^{\hat{r}}{}_{\hat{\varphi}\hat{r}\hat{\varphi}}$$

$$= e^{2\psi}\left[\phi'' + (\phi')^2 - \psi'' + \tfrac{(\phi' - \psi')}{r}\right] \tag{9}$$

The Stress Energy Tensor.

Einstein equations require the stress energy tensor which proportional to the Einstein tensor. In orthonormal basis stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$must have the same algebraic structure as $G_{\hat{\mu}\hat{\nu}}$. the only nonzero components are $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}}$. Using the orthonormal basis these components carry a simple physical interpretation.

i.e. $T_{\hat{t}\hat{t}} = \rho(r)c^2$, $T_{\hat{r}\hat{r}} = -\tau(r)$, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}} = p(r)$,

Where $\rho(r)$ is the energy is the energy density, $\tau(r)$ is the radial tension and $p(r)$ is the pressure measured in the cross-radial direction.

Thus the Einstein field equation , $G_{\hat{\mu}\hat{\nu}} = 8\pi Gc^{-4}T_{\hat{\mu}\hat{\nu}}$ provides the following stress-energy scenario:

$$e^{2\psi}\left(2\psi'' - (\psi')^2 + \tfrac{4\psi'}{r}\right) = 8\pi Gc^{-2}\rho(r) \tag{10}$$

$$e^{2\psi}\left[(\psi')^2 - 2\phi'\psi' + \tfrac{2(\phi' - \psi')}{r}\right] = -8\pi Gc^{-4}\tau(r) \tag{11}$$

$$e^{2\psi}\left[\phi'' + (\phi')^2 - \psi'' + \tfrac{(\phi' - \psi')}{r}\right] = 8\pi Gc^{-4}p(r) \tag{12}$$

Differentiating (11) and using equation (12) one can get the relativistic Euler equation or hydrostatic equation for the material threading the wormhole as

$$\tau' = (\rho c^2 - \tau)\phi' + \left(2\psi' - \frac{2}{r}\right)(p + \tau) \tag{13}$$

Which can be written as

$$p = -\tau + \left(2\,\psi' - \frac{2}{r}\right)^{-1}[\tau' - (\rho c^2 - \tau)\phi'] \tag{14}$$

The Mathematics of Embedding.

Considering a fixed moment of time , $t =$ const., the respective line element is given by $ds^2 = e^{-2\psi(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]$. In three dimensional Euclidean space the embedded surface has equation $z = z(r)$, so that the metric of the surface can be written as

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 e^{-2\psi(r)}(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Comparing we get,

$$\frac{dz}{dr} = \pm\left(e^{-2\psi(r)} - 1\right)^{1/2} \tag{15}$$

To be a wormhole solution , the geometry has a minimum at $r_0 \neq 0$, Where r_0 is the location of throat, at which the embedded surface is vertical ,i.e. $dz/dr \to \infty$. Far from the throat , one may consider the space is asymptotically flat, i.e. $dz/dr \to 0$ as $r \to \infty$.

Here $C(r) \equiv 2\pi r\, e^{-\psi(r)}$ has a minimum at $r_0 \neq 0$ and $dl^2 = e^{-2\psi(r)}dr^2$.

Therefore , $\frac{dC(r)}{dl} = \frac{dC(r)}{dr}\frac{dr}{dl} = 2\pi(1 - r\psi') = 0$ at the throat r_0 and for minimum i.e. flaring out condition is $\frac{d^2C(r)}{dl^2} > 0 \Rightarrow (1 - r\psi')' > 0$ at the throat r_0. (16)

Exotic Matter.

The first two Einstein equations (10) and (11) can combined to give

$$\begin{aligned} 8\pi G c^{-4}(\rho c^2 - \tau) &= 2e^{2\psi}[\psi'' + \frac{\psi'}{r} + \frac{\phi'}{r}(1 - r\psi')] \\ &= 2e^{2\psi}[-\frac{1}{r}(1 - r\psi')' + \frac{\phi'}{r}(1 - r\psi')] \\ &= \frac{2e^{2\psi}}{r}[-(1 - r\psi')' + \phi'(1 - r\psi')] \\ &= -\frac{2e^{2\psi}}{2\pi r}\frac{d^2C(r)}{dl^2} + \frac{2e^{2\psi}}{r}\phi'(1 - r\psi') \end{aligned} \tag{17}$$

At the throat, $(1 - r_0\psi'(r_0)) = 0 \Rightarrow 8\pi G c^{-4}(\rho_0 c^2 - \tau_0) = -\frac{2e^{2\psi(r_0)}}{r_0}\left(1 - r_0\psi'(r_0)\right)' < 0$ by flaring out condition (16) at the throat. This violets the NEC and Morris and Thorne coined matter constrained by this condition as ‘exotic matter’.

TRAVERSABILITY CONDITION.

In constructing traversable wormhole geometries , we will be interested in specific solution by imposing specific traversability conditions. Assume that a traveler of an absurdly advanced civilization beings the trip in a space station in the lower universe, at proper distance $l = -l_1$ and ends up in the upper universe at $l = +l_2$. Furthermore consider that the traveler has a radial velocity $v(r)$, as measured by a static observer positioned at r. One may relate the proper distance travelled dl, radius travelled dr, coordinate time lapse dt and proper time lapse as seen by the traveler $d\tau$ as

$$v(r) = \frac{proper\ distance\ traveled\ by\ traveler\ measured\ by\ observer}{time\ lapse\ measured\ by\ observer}$$

$$= \frac{dl}{e^{\phi(r)}dt}$$

$$= \mp e^{-\phi} e^{-\psi} \frac{dr}{dt} \qquad (18)$$

And $$v\gamma \equiv \frac{v}{[1-(^v/_c)^2]^{1/2}} = \frac{dl}{d\tau} = \mp e^{-\psi} \frac{dr}{d\tau} \qquad (19)$$

It is important to impose certain condition at the space stations

(1) Space is asymptotically flat at the stations , i.e. $e^{\psi} \gg 0$. (automatically)

(2) Gravitational redshift of signals sent from the station to infinity must be small.

(3) The 'acceleration of the gravity' measured at the station $g = -e^{\psi}\phi' c^2$ should be less than Earth's gravitational acceleration, i.e. $|e^{\psi}\phi' c^2| \leq g_{\oplus}$ at $l = -l_1$ and $= +l_2$.

Total Time in a Traversal.

The trip should take a relatively short time , for instance, Morris and Thorne consider 1 year, as measured both by traveler and by people who live in the stations,

$$\Delta\tau = \int_{-l_1}^{l_2} \frac{dl}{v\gamma} \leq 1yr., \qquad (20)$$

$$\Delta t = \int_{-l_1}^{l_2} \frac{dl}{ve^{\phi}} \leq 1yr., \qquad (21)$$

Acceleration Felt by a Traveler.

An important traversability condition is that the acceleration felt by the traveler should not exceed Earth's gravity. By Lorentz transformation between the orthonormal bases of the traveler proper reference frame $\{\boldsymbol{e}_{\hat{0}'}, \boldsymbol{e}_{\hat{1}'}, \boldsymbol{e}_{\hat{2}'}, \boldsymbol{e}_{\hat{3}'}\}$ and static observer frame $\{\boldsymbol{e}_{\hat{t}}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\varphi}}\}$.

$$\boldsymbol{e}_{\hat{0}'} = \gamma \boldsymbol{e}_{\hat{t}} \mp \gamma(v/c)\boldsymbol{e}_{\hat{r}}\ ,\ \boldsymbol{e}_{\hat{1}'} = \gamma(v/c)\boldsymbol{e}_{\hat{t}} \mp \gamma \boldsymbol{e}_{\hat{r}}\ ,\ \boldsymbol{e}_{\hat{2}'} = \boldsymbol{e}_{\hat{\theta}}\ ,\ \boldsymbol{e}_{\hat{3}'} = \boldsymbol{e}_{\hat{\varphi}}\ . \qquad (22)$$

The traveler four-acceleration $a^{\hat{\alpha}'} = u^{\hat{\alpha}'}{}_{;\hat{\beta}'} u^{\hat{\beta}'} c^2$ is the acceleration that his body feels. As four acceleration is always orthogonal to four-velocity, thus $\vec{\boldsymbol{a}}.\boldsymbol{e}_{\hat{0}'} = 0 = a_{\hat{0}'}$ and as traveler moves radially, his acceleration be radial, thus $a_{\hat{2}'} = a_{\hat{3}'} = 0$.

Now $a_t/c^2 = u_{t;\alpha}u^\alpha = \frac{\partial u_t}{\partial r}\,u^r - \Gamma^\beta{}_{tr}u_\beta u^r$ in the (ct, r, θ, φ). Here $u_t = g_{tt}u^t = -e^{2\varphi}(\gamma e^{-\varphi}) = -\gamma e^\varphi$ and $u^r = \mp\, v\gamma e^\psi$.

Again,

$$a_t = \vec{\boldsymbol{a}}.\vec{\boldsymbol{e}_t} = (a\boldsymbol{e}_{\hat{1}'}).(e^\varphi \boldsymbol{e}_{\hat{t}}) = a[\gamma(v/c)\boldsymbol{e}_{\hat{t}} \mp \gamma \boldsymbol{e}_{\hat{r}}].(e^\varphi \boldsymbol{e}_{\hat{t}}) = a\gamma(v/c)e^\varphi g_{tt} = -a\gamma(v/c)e^\varphi.$$

$$\Rightarrow (-\gamma e^\varphi)'.(\mp\, v\gamma e^\psi) = -a\gamma(v/c)e^\varphi$$

$$\Rightarrow a = \mp e^\psi e^{-\varphi}(\gamma e^\varphi)'c^2 = \mp e^{-\varphi}\frac{d}{dl}(\gamma e^\varphi)c^2.$$

Our demand that traveler not feel an acceleration larger than Earth's gravity, i.e.

$$\left|e^{-\varphi}\frac{d}{dl}(\gamma e^\varphi)\right| \le g_\oplus/c^2\,. \tag{23}$$

Tidal Acceleration Felt by a Traveler.

Tidal acceleration felt by a traveler is given by

$$\Delta a^{\widehat{\mu'}} = -c^2 R^{\widehat{\mu'}}{}_{\widehat{\beta'}\widehat{\gamma'}\widehat{\delta'}}\,u^{\widehat{\beta'}}\xi^{\widehat{\mu'}}u^{\widehat{\delta'}}\,, \tag{24}$$

where $u^{\widehat{\alpha'}} = \delta^{\widehat{\alpha'}}_{0'}$ is the traveler four velocity and $\boldsymbol{\xi}$ denote vector separation between two parts of his body and it is purely spatial in the traveler reference frame, i.e. $\boldsymbol{\xi}.\boldsymbol{u} = -\xi^{\widehat{0'}} = 0$. Thus,

$$\Delta a^{\widehat{\iota'}} = -c^2 R^{\widehat{\iota'}}{}_{\widehat{0'}\widehat{k'}\widehat{0'}}\xi^{\widehat{k'}} = -c^2 R_{\widehat{\iota'}\widehat{0'}\widehat{k'}\widehat{0'}}\xi^{\widehat{k'}} \tag{25}$$

Using Lorentz transformation of the Riemannian tensor components between the orthonormal bases of the traveler's frame $\{\boldsymbol{e}_{\widehat{0'}}, \boldsymbol{e}_{\widehat{1'}}, \boldsymbol{e}_{\widehat{2'}}, \boldsymbol{e}_{\widehat{3'}}\}$ and static observer frame $\{\boldsymbol{e}_{\hat{t}}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\varphi}}\}$, the nonzero components of $R_{\widehat{\iota'}\widehat{0'}\widehat{\jmath'}\widehat{0'}}$ are given by

$$R_{\widehat{1'}\widehat{0'}\widehat{1'}\widehat{0'}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = g_{\hat{r}\hat{r}}R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = e^{2\psi}[\phi'' + \phi'(\psi' + \phi')] \tag{26}$$

$$\begin{aligned} R_{\widehat{2'}\widehat{0'}\widehat{2'}\widehat{0'}} = R_{\widehat{3'}\widehat{0'}\widehat{3'}\widehat{0'}} &= \gamma^2 R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + \gamma^2\left(\frac{v}{c}\right)^2 R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = \gamma^2 g_{\hat{\theta}\hat{\theta}}R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + \gamma^2\left(\frac{v}{c}\right)^2 g_{\hat{\theta}\hat{\theta}}R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} \\ &= \gamma^2 e^{2\psi}\left[\left(\frac{v}{c}\right)^2\left(\psi'' + \frac{\psi'}{r}\right) + (\frac{1}{r} - \psi')\phi'\right] \\ &= \gamma^2 e^{2\psi}\left[-\left(\frac{v}{c}\right)^2 \frac{1}{r}(1 - r\psi')' + (\frac{1}{r} - \psi')\phi'\right] \end{aligned} \tag{27}$$

Traversability condition require that tidal acceleration felt by traveler should not exceed Earth's gravitational acceleration. Again let, $|\boldsymbol{\xi}| \sim 2\ m$ (size of the traveler body). Thus,

$$|\Delta\vec{\boldsymbol{a}}| \le g_\oplus \Rightarrow |\,R_{\widehat{1'}\widehat{0'}\widehat{1'}\widehat{0'}}| = \left|e^{2\psi}[\phi'' + \phi'(\psi' + \phi')]\right| \le \frac{g_\oplus}{c^2 \times 2m} \tag{28}$$

And $|R_{\widehat{2'}\widehat{0'}\widehat{2'}\widehat{0'}}| = |R_{\widehat{3'}\widehat{0'}\widehat{3'}\widehat{0'}}| = \left|\gamma^2 e^{2\psi}\left[-\left(\frac{v}{c}\right)^2 \frac{1}{r}(1 - r\psi')' + (\frac{1}{r} - \psi')\phi'\right]\right| \le \frac{g_\oplus}{c^2 \times 2m}$ (29)

WORMHOLE SOLUTION.

The shape function $\psi(r)$ that generate this solution must satisfy the wormhole-shaping conditions. If we take, $\psi(r) = \log\left(\frac{2r - r_0}{2r}\right)$, then $\psi'(r_0) = \frac{1}{r_0}$ which satisfy the condition $\frac{dC(r)}{dl} =$

$\frac{dC(r)}{dr}\frac{dr}{dl} = 2\pi(1 - r\psi') = 0$ at the throat r_0. Again, $\frac{dz}{dr} = \pm\left(e^{-2\psi(r)} - 1\right)^{1/2} \to \pm\sqrt{3}\ as\, r \to r_0$ and $\frac{dz}{dr} \to 0$ as $r \to \infty$, i.e. flaring out condition is satisfied. Again, if we take $\phi = 0$., then

$$\rho(r) = (8\pi G c^{-2})^{-1} e^{2\psi}\left(2\,\psi'' - (\psi')^2 + \frac{4\psi'}{r}\right) \tag{30}$$

$$\tau(r) = -(8\pi G c^{-4})^{-1} e^{2\psi}\left[(\psi')^2 - \frac{2\psi'}{r}\right] \tag{31}$$

$$p(r) = -(8\pi G c^{-4})^{-1} e^{2\psi}\left[\psi'' + \frac{\psi'}{r}\right] \tag{32}$$

In this case, $\psi' = \frac{r_0}{r(2r - r_0)}$ and $\psi'' = -\frac{r_0(4r - r_0)}{r^2(2r - r_0)^2}$. As $r \to \infty$, ψ' and $\psi'' \to 0$, Thus, density $\rho(r)$, tension $\tau(r)$ and pressure $p(r)$ are all asymptotically towards zero as $l \to \pm\infty$.

We shall locate our two space stations at large enough radii, i.e. take $r_1 = r_2 = Lr_0$ (where $\boldsymbol{L}$ is a large positive number) corresponding to $l_1 = l_2 \approx Lr_0$.Now we want to calculate how fast a traveler can traverse the entire wormhole from station 1 at $-l_1$ in the lower universe to station 2 at l_2 in the upper universe. We ignore acceleration at leaving the station 1 and the deceleration upon arriving at station 2 and instead we assume that our traveler maintains constant speed throughout the trip. The acceleration constraint is trivially satisfied since $\phi = 0$ and γ remains constant for the trip. We are thus left with constraint (29) limiting the tidal forces associated with motion through the tunnel

$$\gamma^2\left(\frac{v}{c}\right)^2\left|\frac{e^{2\psi}}{r}(1 - r\psi')'\right| \le \frac{g_\oplus}{c^2 \times 2m}$$

$$\Rightarrow \gamma^2\left(\frac{v}{c}\right)^2\left|e^{2\psi}\left(\psi'' + \frac{\psi'}{r}\right)\right| \le \frac{g_\oplus}{c^2 \times 2m} \tag{33}$$

Substituting the values, we get

$$\gamma^2\left(\frac{v}{c}\right)^2\frac{r_0}{2r^3} \le \frac{g_\oplus}{c^2 \times 2m}$$

.

This constraint is most severe for the smallest radius $r = r_0$ (at the throat)

$$\gamma^2\left(\frac{v}{c}\right)^2 \le \frac{g_\oplus \times {r_0}^2}{c^2 \times 1m}$$

.In the limit that the motion is non relativistic $({}^{v}/{}_{c} \ll 1, \gamma \approx 1)$ we obtain

$$v \le r_0\sqrt{g_\oplus} \tag{34}$$

Correspondingly, the total time lapse for travel from station 1 to station 2 is the same(since, $\gamma \approx 1$ and $\phi = 0$) for clocks ticking in the station and board on the spaceship:

$$\Delta\tau \approx \Delta t \approx \int_{-l_1}^{l_2}\frac{dl}{v} \approx 2 \times \frac{Lr_0}{v} \approx 2 \times \frac{L}{\sqrt{g_\oplus}}. \tag{35}$$

Thus the total time trip through such a tunnel can be made a comfortable hour and velocity can be make large or small according to the throat radius r_0.

CONCLUSION.

In this paper it is completely describe the mathematical structure of wormhole with isotropic coordinates and finding the Euler equation for the wormhole and how flaring out condition imply the exotic matter condition. From the given solution it is seen that energy density of the matter, radial tension and pressure are all asymptotically tends to zero for asymptotically flat surface. It is also seen that traversability is possible for zero redshift function and the total time trip through such a tunnel can be made a comfortable hour and velocity can be make large or small according to the throat radius.

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