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Finite Double Integral Representation For The Polynomial Set $S_n(X, Y)$

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ABSTRACT

In the present paper, an attempt has been made to express a Finite Double Integral representation for the polynomial set $S_n(x, y)$. Many interesting new results may be obtained as particular cases on separating the parameter.

KEYWORD: Appell Function, Generalized Hyper geometric Polynomial, Integral Representation, Orthogonal Polynomial.

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1. INTRODUCTION

We define the generalized hypergeometric polynomial set $S_n(x, y)$ by means of the generating functions,

$$\begin{aligned}
 & e^{\lambda y t} F \left[\begin{matrix} (G_r); \\ \lambda_1 y^{e_1} t^{e_1} \end{matrix} \right] \times F \left[\begin{matrix} (a_p); (A_h); (C_u) \\ \lambda_3 x^{e_3} t, \lambda_2 x^{e_2} y^{-e_2} t^{e_2} \end{matrix} \right] \\
 & \left[\begin{matrix} (H_s); \end{matrix} \right] \left[\begin{matrix} (b_q); (B_k); (D_v) \end{matrix} \right] \\
 & = \sum_{n=0}^{\infty} S_{n_1; e_1; e_2; e_3; (H_s); (b_q); (B_k); (D_v)}^{\lambda; \lambda_1; \lambda_2; \lambda_3; (G_r); (a_p); (A_h); (C_u)}(x, y) t^n \quad \dots (1.1)
 \end{aligned}$$

Where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are real and e_1, e_2, e_3 are positive integers.

The left hand side of (1.1) contains Appell function of two variables in the notation of Burchnall and Chaundy¹

The polynomial set contains a number of parameters, for simplicity, we shall denote

$$S_{n_1; e_1; e_2; e_3; (H_s); (b_q); (B_k); (D_v)}^{\lambda; \lambda_1; \lambda_2; \lambda_3; (G_r); (a_p); (A_h); (C_u)}(x, y) \text{ by } S_n(x, y).$$

Where n denote the order of the polynomial set.

After little simplification (1.1) gives

$$S_n(x, y) = \sum_{\substack{m, m_1, m_2 > 0 \\ m + e_1 m_1 + e_2 m_2 \leq 0}} \frac{\Delta(m_1, m_2)}{(n - m - e_1 m_1 - e_2 m_2)!} \quad \dots (1.2)$$

$$\text{Where } \Delta(m_1, m_2) = \frac{[(a_p)]_{n-m-e_1 m_1 - (e_2-1)m_2}}{[(b_q)]_{n-m-e_1 m_1 - (e_2-1)m_2}}$$

$$\begin{aligned}
 & \times \frac{[(A_h)]_{n-m-e_1 m_1 - e_2 m_2} [(G_r)]_{m_1} [(C_u)]_{m_2}}{[(B_k)]_{n-m-e_1 m_1 - e_2 m_2} [(H_s)]_{m_1} [(D_v)]_{m_2}} \\
 & \times \frac{x^{m_2 e_2} \lambda^m \lambda_1^{m_1} \lambda_2^{m_2} (\lambda_3 x^{e_3})^{n-m-e_1 m_1 - e_2 m_2} y^{m+e_1 m_1 + e_2 m_2}}{m! m_1! m_2! (n - m - e_1 m_1 - e_2 m_2)!}
 \end{aligned}$$

The polynomial set $S_n(x, y)$ happens to the generalization of as many as forty-one orthogonal and non-orthogonal polynomials.

2. NOTATIONS

- (i) $(m) = 1, 2, 3, \dots, m$.
- (ii) $(A_p) = A_1, A_2, A_3, \dots, A_p$.
- (iii) $[(A_p)] = A_1 A_2 A_3 \dots A_p$.
- (iv) $[(A_p)]_n = (A_1)_n (A_2)_n (A_3)_n \dots (A_p)_n$.
- (v) $\Delta(a, b) = \frac{b}{a}, \frac{b+1}{a}, \dots, \frac{b+a-1}{a}$.
- (vi) $\Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b)$.
- (vii) $\Gamma_* \Gamma_*(a+b) = \Gamma(a+b)\Gamma(a+b)$.
- (viii) $K = \frac{[(a_p)]_n [(A_h)]_n (\lambda_3 x^{e_3})^n}{[(b_q)]_n [(B_k)]_n n!}$

3. THEOREM

For $e_2 > 1$, we achieve

$$\begin{aligned}
 S_n(x, y) &= \frac{\Gamma(\alpha + \beta + 1)}{(1-x)^{\alpha+\beta} \Gamma(\alpha)\Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1} \\
 &\times \sum_{m=0}^n \sum_{m_1=0}^{\left[\frac{n-m}{e_1} \right]} \times \frac{[(a_p)]_{n-m-e_1 m_1} [(A_h)]_{n-m-e_1 m_1} [(G_r)]_{m_1}}{[(b_q)]_{n-m-e_1 m_1} [(B_k)]_{n-m-e_1 m_1} [(H_s)]_{m_1}} \\
 &\times \frac{\lambda^m \lambda_1^{m_1} (\lambda_2 x^{e_2})^{n-m-e_1 m_1} y^{m+e_1 m_1}}{m! m_1! (n-m-e_1 m_1)} \\
 &\times F \left[\begin{array}{c} \Delta(e_2; -n+m+e_1 m_1), \Delta(e_2-1; 1-(b_q)-n+m+e_1 m_1) \\ \Delta(e_2; 1-(B_k)-n+m+e_1 m_1), (C_u), \Delta(2e_2; \alpha+\beta+1); \\ \frac{\lambda_2 x^{e_2} \{-e_2\}^{e_2(q-p+h-k+1)} \{-(e_2-1)\}^{(e_2-1)(q-p)}}{(\lambda_3 x^{e_3} y)^{e_2} (1-x)^{2e_2}} \\ \Delta(e_2-1; 1-(a_p)-n+m+e_1 m_1), \\ \Delta(e_2; 1-(A_h)-n+m+e_1 m_1), (D_u); \alpha, \beta; \end{array} \right] dydz \dots
 \end{aligned}
 \tag{3.1}$$

Where $x + y < 1$.

Proof: $I = \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1} \sum_{m=0}^n \sum_{m_1=0}^{\left[\frac{n-m}{e_1} \right]} \times \frac{[(a_p)]_{n-m-e_1 m_1}}{[(b_q)]_{n-m-e_1 m_1}}$

$$\begin{aligned}
 & \times \frac{[(A_h)]_{n-m-e_1 m_1} [(G_r)]_{m_1} \lambda^m \lambda_1^{m_1} (\lambda_2 x^{e_2})^{n-m-e_1 m_1} y^{m+e_1 m_1}}{[(B_k)]_{n-m-e_1 m_1} [(H_s)]_{m_1} m! m_1! (n-m-e_1 m_1)} \\
 & \times F \left[\begin{array}{l} \Delta(e_2; -n+m+e_1 m_1), \Delta(e_2-1; 1-(b_q)-n+m+e_1 m_1), \\ \Delta(e_2; 1-(B_k)-n+m+e_1 m_1), (C_u), \Delta(2e_2; \alpha+\beta+1); \\ \frac{\lambda_2 x^{e_2} \{-e_2\}^{e_2(q-p+h-k+1)} \{-(e_2-1)\}^{(e_2-1)(q-p)}}{(\lambda_3 x^{e_3} y)^{e_2} (1-x)^{2e_2}} \\ \Delta(e_2-1; 1-(a_p)-n+m+e_1 m_1), \\ \Delta(e_2; 1-(A_h)-n+m+e_1 m_1), -(D_u), \alpha, \beta; \end{array} \right] dydz \\
 & = \int_0^{1-x} \int_0^{1-x-y} y^{\alpha+m_2-1} z^{\beta+m_2-1} \sum_{m=0}^n \sum_{m_1=0}^{\lfloor \frac{n-m}{e_1} \rfloor} \times \frac{[(a_p)]_{n-m-e_1 m_1}}{[(b_q)]_{n-m-e_1 m_1}} \\
 & \times \frac{[(A_h)]_{n-m-e_1 m_1} [(G_r)]_{m_1} \lambda^m \lambda_1^{m_1} (\lambda_2 x^{e_2})^{n-m-e_1 m_1} y^{m+e_1 m_1}}{[(B_k)]_{n-m-e_1 m_1} [(H_s)]_{m_1} m! m_1! (n-m-e_1 m_1)} \\
 & \times \frac{\Delta_{m_2} [e_2; -n+m+e_1 m_1] \Delta_{m_2} [e_2-1; 1-(b_q)-n+m+e_1 m_1]}{\Delta_{m_2} [e_2-1; 1-(a_p)-n+m+e_1 m_1] \Delta_{m_2} [e_2; 1-(A_h)-n+m+e_1 m_1]} \\
 & \times \frac{\Delta_{m_2} [e_2; 1-(B_k)-n+m+e_1 m_1] [(C_u)]_{m_2} \Delta_{m_2} (2e_2; \alpha+\beta+1)}{[(D_u)]_{m_2} m_2! (\alpha)_{m_2} (\beta)_{m_2}} \\
 & \times \frac{(\lambda_2 x^{e_2})^{m_2} (-e_2)^{e_2(q-p+h-k+1)m_2} \{-(e_2-1)\}^{(e_2-1)(q-p)} dydz}{(1-x)^2 (\lambda_3 x^{e_3} y)^{e_2 m_2}} \\
 & = \sum_{m=0}^n \sum_{m_1=0}^{\lfloor \frac{n-m}{e_1} \rfloor} \sum_{m_2=0}^{\lfloor \frac{n-m-e_1 m_1}{e_2} \rfloor} \times \frac{[(a_p)]_{n-m-e_1 m_1} [(A_h)]_{n-m-e_1 m_1}}{[(b_q)]_{n-m-e_1 m_1} [(B_k)]_{n-m-e_1 m_1}} \\
 & \times \frac{[(G_r)]_{m_1} \lambda^m \lambda_1^{m_1} (\lambda_2 x^{e_2})^{n-m-e_1 m_1} y^{m+e_1 m_1}}{[(H_s)]_{m_1} m! m_1! (n-m-e_1 m_1)} \\
 & \times \frac{\Delta_{m_2} [e_2; -n+m+e_1 m_1] \Delta_{m_2} [e_2-1; 1-(b_q)-n+m+e_1 m_1]}{\Delta_{m_2} [e_2-1; 1-(a_p)-n+m+e_1 m_1]}
 \end{aligned}$$

$$\begin{aligned} & \frac{\Delta_{m_2} [e_2; 1 - (B_k) - n + m + e_1 m_1] [(C_u)]_{m_2}}{\Delta_{m_2} [e_2; 1 - (A_h) - n + m + e_1 m_1] [(D_u)]_{m_2}} \\ & \times \frac{\Delta_{m_2} [2e_2; \alpha + \beta + 1] (\lambda_2 x^{e_2})^{m_2} (-e_2)^{e_2(q-p+h-k+1)m_2}}{(\alpha)_{m_2} (\beta)_{m_2} (1-x)^{2m_2} (\lambda_3 x^{e_3} y)^{e_2 m_2}} \\ & \times \frac{(1-x)^{\alpha+\beta+2m_2} \Gamma(\alpha+m_2) \Gamma(\beta+m_2) \{(e_2-1)\}^{(e_2-1)(q-p)m_2}}{\Gamma(\alpha+\beta+1+2m_2)} \\ & = \frac{(1-x)^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta+1)} S_n(x, y) \end{aligned}$$

On using Orthogonal Polynomials by Gabor Szego²

$$\int_0^{1-x} \int_0^{1-x-y} y^{m-1} z^{n-1} dy dz = \frac{(1-x)^{m+n} \Gamma(m) \Gamma(n)}{\Gamma(m+n+1)}$$

Where $x + y < 1$

Particular Cases of (3.1)

Separating the term corresponding to $\lambda = 0$ and putting $r = 0 = s = \lambda_1$ in (3.1), we obtain a number of results on specializing the remaining parameters :

(i) Hermit polynomials:

On Putting $p = 0 = q = h = k = u = v; m = 1 = m_1 = e_1 = e_3 = \lambda; \lambda_3 = 1 = e_2; \lambda_2 = -1, y = x$, We get

$$\begin{aligned} H_n(x) &= \frac{(2x)^n \Gamma(\alpha + \beta + 1)}{n!(1-x)^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1} \\ & \times F \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \Delta(2; \alpha + \beta + 1); \\ \alpha, \beta; \end{matrix} \frac{-1}{(1-x)^2} \right] dy dz \end{aligned}$$

(ii) Legendre polynomials:

If we take $p = 0 = q = h = k = u; v = 1 = m = m_1 = e_1 = \lambda_2 = e_3; D_1 = 1; \lambda_3 = 1;$

$y = 2x, e_2 = 2$, and $\frac{x}{\sqrt{x^2 - 1}}$ for x , we get

$$P_n(x) = \frac{x^n \Gamma(\alpha + \beta + 1)}{n!(1-x)^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1}$$

$$\times F \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \Delta(2; \alpha + \beta + 1); \\ \frac{x^2 - 1}{x^2(1-x)^2} \\ 1, \alpha, \beta; \end{matrix} \right] dydz$$

ii. Jackson polynomials: On taking $p=0=q=h=k=u=v; e_2=2, \lambda_3=4; \lambda_2=-4, e_2=4, e_3=1=\lambda=\lambda_1; y=x$

We achieve

$$\mathbb{H}_n(x) = \frac{2^{2n} x^n \Gamma(\alpha + \beta + 1)}{n!(1-x)^{\alpha+\beta} \Gamma(\alpha)\Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1}$$

$$\times F \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \Delta(2; \alpha + \beta + 1); \\ \frac{-1}{x^2(1-x)^2} \\ \alpha, \beta; \end{matrix} \right] dydz$$

(iv) If we set $p=0=q=h=k; u=1=v=e_3; y=x; e_2=m; \lambda_2=\mu; \lambda_3=v; C_1=a_r, D_1=b_s,$

We have

$$A_n(x) = \frac{(vx)^n \Gamma(\alpha + \beta + 1)}{n!(1-x)^{\alpha+\beta} \Gamma(\alpha)\Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1}$$

$$\times F \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \Delta(2; \alpha + \beta + 1), (a_r); \\ \mu \left(\frac{-m}{vx} \right)^m \frac{1}{(1-x)^2} \\ \alpha, \beta; (b_s) \end{matrix} \right] dydz$$

Where $A_n(x)$ are the generalized by Panda³

(v) On making the substitution $p=0=q=h=k=u=v; e_3=1=\lambda_3; \lambda_2=h, y=x,$

We have

$$g_n^m(x, h) = \frac{x^n \Gamma(\alpha + \beta + 1)}{n!(1-x)^{\alpha+\beta} \Gamma(\alpha)\Gamma(\beta)} \int_0^{1-x} \int_0^{1-x-y} y^{\alpha-1} z^{\beta-1} \times F \left[\begin{matrix} \Delta(m; -n), \Delta(2; \alpha + \beta + 1), \\ \frac{h}{(1-x)^2} \left(\frac{-m}{x} \right)^m \\ \alpha, \beta; \end{matrix} \right] dydz$$

Where $g_n^m(x, h)$ are the generalized polynomials defined by Gould-Hopper⁴

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