Research article Available online www.ijsrr.org ISSN: 2279–0543

International Journal of Scientific Research and Reviews

Analysis of Complex Continued Fractions of The Solution of The Equation $kx^2 - nx + n = 0$ Where n Is An Odd Integer And $k = \left[\frac{n}{2}\right]$

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ABSTRACT

In this paper we identify the complex continued fractions of the solution of the equation $kx^2 - nx + n = 0$ where n is an odd integer and $k = \left[\frac{n}{2}\right]$. The analysis of the complex continued fractions of the above equations when $n > 3, n = 3$ & $k = n = 1$ are found and their comparisons are made.

KEYWORDS: Continued fraction, Finite complex continued fraction, Infinite complex continued fraction, Periodic complex continued fraction, Gaussian integer, Euclidean algorithm, Quadratic irrational.

SUBJECT CLASSIFICATION: MSC 11A05, 11A55, 11J70, 11Y65, 30B70, 40A15.

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I. BASIC DEFINITIONS [1, 2]

Let $C=\{z=x+iy\,/\,x,y\in R\}$ be the set of all complex numbers. Where x and y are called the real and imaginary parts of z. These are denoted by $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ respectively. The conjugate of the complex number z is denoted by $\bar{z}=x-iy$ and the modulus of z is denoted by $|z|=\sqrt{x^2+y^2}$. Let $Z(i)=\{x+iy\,/\,x,y\in Z\}$ be the set of all Gaussian integers. Let $Q(i)=\left\{\frac{p}{q}\,/\,p,q\in Z(i)\,and\,q\neq 0\right\}$ be the set of all complex rational numbers. So that $Z(i)\subset Q(i)\subset C$ .Let $z\in C$ then z is rational if $z\in Q(i)$, z is irrational if $z\in C\setminus Q(i)$ and z is quadratic irrational if z is irrational and there exist $A,B,C\in Z(i)$ such that $Az^2+Bz+C=0$ and $A\neq 0$.

Greatest common divisor of any two complex numbers: [3, 4, 5]

Let $z_1,z_2\in Z(i), z_2\neq 0$. Then the greatest common divisor of z_1 and z_2 is denoted by $\gcd(z_1,z_2)$ and it is defined to be a Gaussian integer g such that $g\,|\,z_1$ and $g\,|\,z_2$ and there is no Gaussian integer h such that $|h|>|g|$ that divides both z_1 and z_2. In this case we write $\gcd(z_1,z_2)=g$

If $x,y\in R$ then the floor function of x and y are denoted by $[x]$ and $[y]$ respectively. So that $x-[x]<1$ and $y-[y]<1$. Let $z\in C$. Then z is a quadratic irrational if and only is $z=\frac{p+q\sqrt{r}}{s}$ for some $p,q,r,s\in Z(i), q\neq 0, s\neq 0$ and r is not a perfect square.

II. INTRODUCTION

An expression of the form

$$a_0+\cfrac{e_0}{a_1+\cfrac{e_1}{a_2+\cfrac{e_2}{a_3+\cfrac{e_3}{\ddots+\cfrac{e_n}{a_n+\ddots}}}}}$$

Where $a_0,a_1,a_2,a_3,\cdots$ are in $Z(i)$ and $e_i{}'s$ are units of complex numbers. Therefore $e_k\in\{1,-1,i,-i\}, k=1,2,3,...$ is known as a complex continued fraction.[6,7,8]

The complex continued fraction is commonly expressed as $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots$.

The quantities $a_0,\frac{e_1}{a_1},\frac{e_2}{a_2},\frac{e_3}{a_3},\cdots$ are called the elements of the complex continued fraction.

In a finite complex continued fraction the number of elements are finite. Where as an infinite continued fraction have infinite number of quantities.

Therefore $a_0 + \frac{e_1}{a_1 +} \frac{e_2}{a_2 +} \frac{e_3}{a_3 +} \cdots \frac{e_n}{a_n}$ is known as finite complex continued fraction and an expression $a_0 + \frac{e_1}{a_1 +} \frac{e_2}{a_2 +} \frac{e_3}{a_3 +} \cdots \frac{e_n}{a_n +} \frac{e_{n+1}}{a_{n+1} +} \cdots$ is known as an infinite complex continued fraction. In a finite or infinite complex continued fraction $a_0, a_1, a_2, a_3, \cdots$ are called the partial quotients where as $e_1, e_2, e_3, \cdots$ are known as the partial numerators.

The length of a finite complex continued fraction $a_0 + \frac{e_1}{a_1 +} \frac{e_2}{a_2 +} \frac{e_3}{a_3 +} \cdots \frac{e_n}{a_n}$ is $n+1$ and the length of a infinite continued fraction $a_0 + \frac{e_1}{a_1 +} \frac{e_2}{a_2 +} \frac{e_3}{a_3 +} \cdots \frac{e_n}{a_n +} \frac{e_{n+1}}{a_{n+1} +} \cdots$ is ∞.The value of the finite complex continued fraction is denoted as $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n} \right]$.

2.1 Convergent of complex continued fraction:[6,7,8]

The successive convergent of the complex continued fractions are $[a_0]$, $\left[a_0, \frac{e_1}{a_1} \right]$, $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2} \right]$ and so on and they are denoted by $c_0 = \frac{p_0}{q_0}$, $c_1 = \frac{p_1}{q_1}$, $c_2 = \frac{p_2}{q_2}, \ldots$ In general the nth convergent is denoted by $c_n = \frac{p_n}{q_n} = \left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n} \right]$. Where p_i's and q_i's are called the numerators and denominators of the complex continued fraction.

2.2 Properties of complex continued fraction :[6,7,8]

Let $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n}, \cdots \right]$ be an infinite complex continued fraction. We inductively define two infinite sequences p_k and q_k where $k \geq 1$ by

$$p_0 = a_0 \qquad p_{-1} = 1 \qquad q_0 = 1 \qquad q_{-1} = 0$$

$$p_k = a_k\, p_{k-1} + e_k\, p_{k-2}, k \geq 0 \qquad q_k = a_k\, q_{k-1} + e_k\, q_{k-2}, k \geq 0$$

III. ALGORITHMS AND THEOREMS

3.1 Algorithm of complex continued fraction:[9]

Let $x \in C$ .Suppose we wish to find continued fraction expansion of x, take $x_0 = \text{Re}(x) + i\,\text{Im}(x)$.

Let $a = \mathrm{Re}(x) - [\mathrm{Re}(x)]$ and $b = \mathrm{Im}(x) - [\mathrm{Im}(x)]$ so that $a < 0$ and $b < 0$ .Now $[x_0] = [\mathrm{Re}(x)] + i[\mathrm{Im}(x)] + \alpha$, a floor function. Where $\alpha = \begin{cases} 0 & if\ a+b<0 \\ 1 & if\ a+b \geq 1 with\ a \geq b \\ i & if a+b \geq 1 with\ a < b \end{cases}$

Set $a_0 = [x_0]$. Then $x_1 = \dfrac{1}{x_0 - [x_0]}$ and $[x_1] = [\mathrm{Re}(x_1)] + i[\mathrm{Im}(x_1)] + \alpha$.

Again set $a_1 = [x_1]$. Then $x_2 = \dfrac{1}{x_1 - [x_1]}$ and $[x_2] = [\mathrm{Re}(x_2)] + i[\mathrm{Im}(x_2)] + \alpha$, $a_2 = [x_2]$ .Then $x_3 = \dfrac{1}{x_2 - [x_2]}$ and $[x_3] = [\mathrm{Re}(x_3)] + i[\mathrm{Im}(x_3)] + \alpha$, $a_3 = [x_3]$ .... $a_{k-1} = [x_{k-1}]$.

Then $x_k = \dfrac{1}{x_{k-1} - [x_{k-1}]}$ and $[x_k] = [\mathrm{Re}(x_k)] + i[\mathrm{Im}(x_k)] + \alpha$ .and $a_k = [x_k]$.

The algorithm terminates if the complex continued fraction is finite otherwise it is no terminating.

3.2 Euclid's algorithm for complex continued fraction:[4]

Let $u_0, u_1 \in Z(i)$. Then we try to find the gcd (u_0, u_1) by using Euclid's algorithm.

Divide u_0 by u_1 if $N(u_0) > N(u_1)$, where N is the norm of the complex number.

Therefore $\dfrac{u_0}{u_1} = a_0 + \dfrac{u_2}{u_1}$ where $a_0 \in Z(i)$ is the quotient and $u_2 \in Z(i)$ is the remainder.

Again $\dfrac{u_1}{u_2} = a_1 + \dfrac{u_3}{u_2}$ where $a_1 \in Z(i)$ is the quotient and $u_3 \in Z(i)$ is the remainder.

And $\dfrac{u_2}{u_3} = a_2 + \dfrac{u_4}{u_3}$ where $a_2 \in Z(i)$ is the quotient and $u_4 \in Z(i)$ is the remainder....

$\dfrac{u_{n-1}}{u_n} = a_{n-1} + \dfrac{u_{n+1}}{u_n}$ where $a_{n-1} \in Z(i)$ is the quotient and $u_{n+1} \in Z(i)$ is the remainder.

And $\dfrac{u_n}{u_{n+1}} = a_n$ where $a_n \in Z(i)$ is the quotient and 0 is the remainder.

Here $\dfrac{u_j}{u_{j+1}} \in C, j = 0,1,2,...n$ Where a_i's $\in Z(i), i = 0,1,2,..n$ are the quotients and u_j's $\in Z(i), j = 2,3,..n+1$ are the remainders. These quotients are treated as the partial quotients of the $\dfrac{u_0}{u_1}$, the ratio of the complex numbers.

Therefore the complex continued fraction of $\dfrac{u_0}{u_1} = [a_0, a_1, ..., a_n]$.

Note:

1. $a_0 = \mathrm{Re}\left[\frac{u_0}{u_1}\right] + i\,\mathrm{Im}\left[\frac{u_0}{u_1}\right] + \alpha$ where α is a floor function and

$$\alpha = \begin{cases} 0 & if\ a+b<0 \\ 1 & if\ a+b \geq 1\, with\ a \geq b \\ i & if a+b \geq 1\, with\ a<b \end{cases}, \quad a = \mathrm{Re}\left(\frac{u_0}{u_1}\right) - \left[\frac{u_0}{u_1}\right] \text{ and } b = \mathrm{Im}\left(\frac{u_0}{u_1}\right) - \left[\frac{u_0}{u_1}\right].$$

The successive quotients are found by the same procedure.

2. Remainders are found by using the relations $u_{j+1} = u_{j-1} - a_{j-1}u_j$, $j = 1,2,...n$.

Example 3.2.1 : $\gcd(3+13i,\ 4+3i)$ Since $N(3+13i) = 178 > 25 = N(4+3i)$, divide $(3+13i)$ by $(4+3i)$.

Therefore $\frac{3+13i}{4+3i} = (2+2i) + \frac{1-i}{4+3i} \Rightarrow \frac{4+3i}{1-i} = (3i) + \frac{1}{1-i} \Rightarrow \frac{1-i}{1} = (1-i).$

Hence $\gcd(3+13i,\ 4+3i) = 1$.

Example 3.2.2 : $\gcd(7-61i,\ 26-36i)$ Since $N(7-61i) = 3770 > 1972 = N(26-36i)$, divide $(7-61i)$ by $(26-36i)$.

Therefore $\frac{7-61i}{26-36i} = (1-i) + \frac{17+i}{26-36i} \Rightarrow \frac{26-36i}{17+i} = (1-2i) + \frac{7-3i}{17+i} \Rightarrow \frac{17+i}{7-3i} = (2+i)$

Hence $\gcd(7-61i,\ 26-36i) = 7-3i$.

Definition:**[3]**

The function $F_f : C \to Z(i)$ is said to be a floor function if $|F(z) - Z| < 1$ for every $Z \in C$.

The function $F_s : C \to \{-1, 1, -i, i\}$ is said to be a sign function.

3.3 Complex continued fraction algorithm (J.O. Shalit,1979):[10]

If $F_f : C \to Z(i)$ is a floor function and $F_s : C \to \{-1, 1, -i, i\}$ is a sign function together with the following sequence is known as a complex continued fraction algorithm.

(i) $Z_0 = Z$

(ii) $a_n = F_f(Z_n)$

(iii) $e_{n+1} = F_s(Z_n)$

(iv) $Z_{n+1} = \frac{e_{n+1}}{Z_n - a_n}$

3.4 Theorem

If $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_j}{a_j}\right] = \left[b_0, \frac{f_1}{b_1}, \frac{f_2}{b_2}, \frac{f_3}{b_3}, \cdots \frac{f_n}{b_n}\right]$ where these finite complex continued fractions are simple if $\frac{e_j}{a_j} > 1$ and $\frac{f_n}{b_n} > 1$ then $j = n$ and $a_i = b_i$, $e_i = f_i$ for $i = 0,1,2,\ldots n$.

Proof:

Let $y_i = \left[b_i, \frac{f_{i+1}}{b_{i+1}}, \frac{f_{i+2}}{b_{i+2}}, \frac{f_{i+3}}{b_{i+3}}, \cdots \frac{f_n}{b_n}\right]$, $y_i = \frac{p_i}{q_i}$

$$= b_i + \frac{f_{i+1}}{\left[b_{i+1}, \frac{f_{i+2}}{b_{i+2}}, \frac{f_{i+3}}{b_{i+3}}, \cdots \frac{f_n}{b_n}\right]} = b_i + \frac{f_{i+1}}{y_{i+1}}.$$

Therefore we have $y_i > b_i$ and $y_i > 1$ for $i = 0,1,2,\ldots n-1$ and $y_n = b_n > 1$.

Also $b_i = \lfloor y_i \rfloor$ for all i, $0 \le i \le n$.

Since $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_j}{a_j}\right] = \left[b_0, \frac{f_1}{b_1}, \frac{f_2}{b_2}, \frac{f_3}{b_3}, \cdots \frac{f_n}{b_n}\right]$, we write $y_0 = x_0$.

For that take $x_i = \frac{p_i'}{q_i'} \Rightarrow x_{i+1} > 1$ for all $i \ge 0$. So that $a_i = \lfloor x_i \rfloor$ $0 \le i \le j$.

Therefore $b_0 = \lfloor y_0 \rfloor = \lfloor x_0 \rfloor = a_0$.

Also $x_i = \left[a_i, \frac{e_{i+1}}{a_{i+1}}, \frac{e_{i+2}}{a_{i+2}}, \frac{e_{i+3}}{a_{i+3}}, \cdots \frac{e_j}{a_j}\right]$, $0 \le i \le j$

$$= a_i + \frac{e_{i+1}}{\left[a_{i+1}, \frac{e_{i+2}}{a_{i+2}}, \frac{e_{i+3}}{a_{i+3}}, \cdots \frac{e_j}{a_j}\right]}$$

$$= a_i + \frac{e_{i+1}}{x_{i+1}} \Rightarrow \frac{e_{i+1}}{x_{i+1}} = x_i - a_i = y_i - b_i = \frac{f_{i+1}}{y_{i+1}}.$$

Therefore $\frac{e_{i+1}}{x_{i+1}} = \frac{f_{i+1}}{y_{i+1}}, i = 0,1,2,\ldots n-1$. So that $e_{i+1} = f_{i+1}$ and $x_{i+1} = y_{i+1}$ and $\lfloor x_{i+1} \rfloor = \lfloor y_{i+1} \rfloor \Rightarrow a_{i+1} = b_{i+1}$ for $i = 0,1,2,\ldots n-1$.

Next to prove that $j = n$.

Case (i): $j < n$.

Since $x_j = y_j \Rightarrow \lfloor x_j \rfloor = \lfloor y_j \rfloor \Rightarrow a_j = b_j$.

Using Euclid's algorithm we get $\frac{p'_j}{q'_j} = x_j = a_j$. Which is a contradiction to $y_j > b_j$.

Case (ii): If $j > n$ then we get the same contradiction.

So that $j = n$.

3.5 Theorem

If $z \in Q(i)$ then the continued fraction of z is finite.

Proof:

$$\text{If } x_n = \frac{p'_n}{q'_n} \text{ then } x_{n+1} = \frac{e_{n+1}}{x_n - a_n} = \frac{f_{n+1}}{\frac{p'_n}{q'_n} - a_n} = \frac{f_{n+1} q'_n}{p'_n - a_n q'_n} = \frac{p'_{n+1}}{q'_{n+1}}.$$

Therefore $x_{n+1} = \frac{p'_{n+1}}{q'_{n+1}} = \frac{f_{n+1} q'_n}{q'_{n+1}}$ for every $n \in N$.

As $|x_{n+1}| > 1$ and $|f_{n+1}| = 1$, $|q'_{n+1}| < |q'_n|$ for all $n \in N$.

As $q'_n \in Z(i)$ there exist $N \in N$ such that $q'_{N+1} = 0$.

If $q'_{N+1} = 0 \Rightarrow p'_N - q'_N a_N = 0 \Rightarrow \frac{p'_N}{q'_N} - a_N = 0 \Rightarrow x_N - a_N = 0$.

Therefore Euclidean algorithm terminates after N transformations.

This completes the proof of the theorem.

3.6 Theorem

If $z \in C \setminus Q(i)$ then the continued fraction of z is infinite.

Proof:

Every rational complex number represents a finite complex continued fraction.

Also in a finite continued fraction $x_n - a_n = 0$ where $a_n = [x_n]$.

If x_0 is irrational then $x_0 - a_0 \neq 0$ or $x_0 - [x_0] \neq 0$ or $x_0 - F_l(x_0) \neq 0$.

Suppose x_k is irrational then $x_k - F_l(x_k) \neq 0$.

Consequently for every $n \in N$, $x_n - a_n \neq 0$. So that continued fraction is infinite.

This completes the proof.

In this paper we try to find the solution of the quadratic equation $kx^2 - nx + n = 0$ where $k = \left[\frac{n}{2}\right]$ and n is odd. Since $n^2 - 4kn < 0$, the roots of the equation are complex. Also complex roots are occurs in pair and continued fraction of these roots are infinite.

If α_1 and α_2 are the complex roots of the equation $kx^2 - nx + n = 0$ then their continued fractions are

$[c_0, c_1, c_2, ...]$ and $[d_0, d_1, d_2, ...]$ respectively. Where c_i's and d_i's $\in Z(i)$.

Here the continued fractions of α_1 and α_2 are calculated by using either using Euclidean algorithm or complex continued fraction algorithm and comparison of their continued fractions are made.

IV. ILLUSTRATIONS

4.1 Illustration

Find the complex continued fraction of the roots of the quadratic equation $4x^2 - 9x + 9 = 0$.

Solution

Roots of the above quadratic equation are $\alpha_1 = 1.1250 + 0.9922i$ and $\alpha_2 = 1.1250 - 0.9922i$

***Case (i):* Continued fraction of the root** $\alpha_1 = 1.1250 + 0.9922i$

Take $x_0 = 1.1250 + 0.9922i$ Then $a = 1.1250 - (1) = 0.1250$ and $b = 0.9922 - (0) = 0.9922$

Therefore $[x_0] = 1 + 0 + i = 1 + i \Rightarrow a_0 = 1 + i$.

Now $x_1 = \dfrac{1}{[(1.1250 + 0.9922i) - (1+i)]} \Rightarrow x_1 = 7.9690 + 0.4973i$

Round off the imaginary part to 0.5 , $x_1 = 7.9690 + 0.5i$

Then $a = 7.9690 - (7) = 0.9690$ and $b = 0.5 - (0) = 0.5$

Therefore $[x_1] = 7 + 0 + 1 = 8 \Rightarrow a_1 = 8$.

Again $x_2 = \dfrac{1}{[(7.9690 + 0.5i) - (8)]} \Rightarrow x_2 = -0.1235 - 1.9923i$

Then $a = -0.1235 - (-1) = 0.8765$ and $b = -1.9923 - (-2) = 0.0077$

Therefore $[x_2] = -1 - 2i + 0 = -1 - 2i \Rightarrow a_2 = -1 - 2i$.

$x_3 = \dfrac{1}{[(-0.1235 - 1.9923i) - (-1 - 2i)]} \Rightarrow x_3 = 1.1408 - 0.0100i$

Then $a = 1.1408 - (1) = 0.1408$and $b = -0.0100 - (-1) = 0.9900$

Therefore $[x_3] = 1 - i + i = 1 \Rightarrow a_3 = 1$.

$x_4 = \dfrac{1}{[(1.1408 - 0.0100i) - (1)]} \Rightarrow x_4 = 7.0666 + 0.5i$

Then $a = 7.0666 - (7) = 0.0666$ and $b = 0.5 - (0) = 0.5$

Therefore $[x_4] = 7 + 0 + 0 = 7 \Rightarrow a_4 = 7$.

$$x_5 = \frac{1}{[(7.0666+0.5i)-(7)]} \Rightarrow x_5 = 0.2618-1.9651i$$

Then $a = 0.2618-(0) = 0.2618$ and $b = -1.9651-(-2) = 0.0349$

Therefore $[x_5] = 0-2i+0 = -2i \Rightarrow a_5 = -2i$.

$$x_6 = \frac{1}{[(0.2618-1.9651i)-(-2i)]} \Rightarrow x_6 = 3.7530-0.50i$$

Then $a = 3.7530-(3) = 0.7530$ and $b = -0.5-(-1) = 0.5$

Therefore $[x_6] = 3-i+1 = 4-i \Rightarrow a_6 = 4-i$.

$$x_7 = \frac{1}{[(3.7530-0.50i)-(4-i)]} \Rightarrow x_7 = -0.7942-1.6077i$$

Then $a = -0.7942-(-1) = 0.2058$ and $b = -1.6077-(-2) = 0.3923$

Therefore $[x_7] = -1-2i+0 = -1-2i \Rightarrow a_7 = -1-2i$.

Case (ii): Continued fraction of the root $\alpha_1 = 1.1250-0.9922i$

Take $x_0 = 1.1250-0.9922i$

Then $a = 1.1250-(1) = 0.1250$ and $b = -0.9922-(-1) = 0.0078$

Therefore $[x_0] = 1-i+0 = 1-i \Rightarrow a_0 = 1-i$.

Now $x_1 = \frac{1}{[(1.1250-0.9922i)-(1-i)]} \Rightarrow x_1 = 7.9690-0.4973i$

Round off the imaginary part to 0.5 ,

$x_1 = 7.9690-0.5i$

Then $a = 7.9690-(7) = 0.9690$ and $b = -0.5-(-1) = 0.5$

Therefore $[x_1] = 7+-i+1 = 8-i \Rightarrow a_1 = 8-i$.

Again $x_2 = \frac{1}{[(7.9690-0.5i)-(8-i)]} \Rightarrow x_2 = -0.1235-1.9923i$

Then $a = -0.1235-(-1) = 0.8765$ and $b = -1.9923-(-2) = 0.0077$

Therefore $[x_2] = -1-2i+0 = -1-2i \Rightarrow a_2 = -1-2i$.

$$x_3 = \frac{1}{[(-0.1235-1.9923i)-(-1-2i)]} \Rightarrow x_3 = 1.1408-0.0100i$$

Then $a = 1.1408-(1) = 0.1408$ and $b = -0.0100-(-1) = 0.9900$

Therefore $[x_3] = 1-i+i = 1 \Rightarrow a_3 = 1$.

$$x_4 = \frac{1}{[(1.1408 - 0.0100i) - (1)]} \Rightarrow x_4 = 7.0666 + 0.5i$$

Then $a = 7.0666 - (7) = 0.0666$ and $b = 0.5 - (0) = 0.5$

Therefore $[x_4] = 7 + 0 + 0 = 7 \Rightarrow a_4 = 7$.

$$x_5 = \frac{1}{[(7.0666 + 0.5i) - (7)]} \Rightarrow x_5 = 0.2618 - 1.9651i$$

Then $a = 0.2618 - (0) = 0.2618$ and $b = -1.9651 - (-2) = 0.0349$

Therefore $[x_5] = 0 - 2i + 0 = -2i \Rightarrow a_5 = -2i$.

$$x_6 = \frac{1}{[(0.2618 - 1.9651i) - (-2i)]} \Rightarrow x_6 = 3.7530 - 0.50i$$

Then $a = 3.7530 - (3) = 0.7530$ and $b = -0.5 - (-1) = 0.5$

Therefore $[x_6] = 3 - i + 1 = 4 - i \Rightarrow a_6 = 4 - i$.

$$x_7 = \frac{1}{[(3.7530 - 0.50i) - (4 - i)]} \Rightarrow x_7 = -0.7942 - 1.6077i$$

Then $a = -0.7942 - (-1) = 0.2058$ and $b = -1.6077 - (-2) = 0.3923$

Therefore $[x_7] = -1 - 2i + 0 = -1 - 2i \Rightarrow a_7 = -1 - 2i$.

From case(i) and case (ii) we get the complex continued fraction of the given equation are

$\alpha_1 = [1 + i, 8, -1 - 2i, 1, 7, -2i, 4 - i, -1 - 2i, \cdots]$ and

$\alpha_2 = [1 - i, 8 - i, -1 - 2i, 1, 7, -2i, 4 - i, -1 - 2i, \cdots]$

4.2 Illustration

Find the complex continued fraction of the roots of the quadratic equation $11x^2 - 23x + 23 = 0$.

Solution

Roots of the above quadratic equation are $\alpha_1 = 1.0455 + 0.9990i$ and $\alpha_2 = 1.0455 - 0.9990i$

Similar to case (i) and case (ii) of previous illustration we find that the complex continued fraction of the given equation are $\alpha_1 = [1 + i, 22, -1 - 2i, 1, 7, -1 - 2i, -2i, ,\cdots]$ and $\alpha_2 = [1 + i, 22 - i, -1 - 2i, 1, 7, -1 - 2i ,-2i, \cdots]$

4.3 Illustration

Find the complex continued fraction of the roots of the quadratic equation $x^2 - 3x + 3 = 0$.

Solution

Roots of the above quadratic equation are $\alpha_1 = 1.50 + 0.8660i$ and $\alpha_2 = 1.50 - 0.8660i$

The complex continued fraction of the roots of the given equation are $\alpha_1 = [1+i, 2, \overline{-1-2i, 2-i}]$ and $\alpha_2 = [1-i, \overline{2-i, -1-2i}]$

4.4 Illustration

Find the complex continued fraction of the roots of the quadratic equation $x^2 - x + 1 = 0$.

Roots of the above quadratic equation are $\alpha_1 = 0.50 + 0.8660i$ and $\alpha_2 = 0.50 - 0.8660i$

Solution

The complex continued fraction of the roots of the given equation are $\alpha_1 = [i, 2, \overline{-1-2i, 2-i}]$ and $\alpha_2 = [-i, \overline{2-i, -1-2i}]$

V. CONCLUSION

From the above illustrations we conclude that the complex continued fraction the quadratic equation of the form $kx^2 - nx + n = 0$ where $k = \left[\frac{n}{2}\right]$ and n is odd and >3 are $\alpha_1 = [c_0, c_1 = 2k, c_2, c_3, c_4, \ldots]$ and $\alpha_2 = [\overline{c_0}, c_1 = 2k - i, c_2, c_3, c_4, \ldots]$. If $n = 3$ then $\alpha_1 = [1+i, 2, \overline{-1-2i, 2-i}]$ and $\alpha_2 = [1-i, \overline{2-i, -1-2i}]$. If $n = 1$ and $k = n$ then $\alpha_1 = [i, 2, \overline{-1-2i, 2-i}]$ and $\alpha_2 = [-i, \overline{2-i, -1-2i}]$. Hence If $n > 3$ the complex continued fraction of the roots of the above quadratic equation are infinite where as if $n = 3$ and when $n = k = 1$ the complex continued fractions are periodic.

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