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Cosmological Models of the Universe

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ABSTRACT

Cosmological models are the mathematical description of the dynamics and evolution of the universe over time. They are based on direct observations and can be used to make predictions that can be confirmed by further investigations. The universe is observed to be approximately homogenous and isotropic over subsequently large spatial distances of the scale of over hundred mega parsecs. This isotropic and homogenous universe is capable of contracting and expanding depending on the parameters like curvature coefficient k, scale factor a (t) and radius of curvature R_{0}. We study the stages of evolution of the universe along with the dynamics using Freidman's equations and solving them for matter, radiation and dark energy dominated each for open k=-1, closed k=1 and flat k=0, to obtain the value of scale factor and time as the evolution proceeds through stages where various types of energy become dominated, as the previous one decays. A graph plot built using 'Gnu plot', showing the age of the universe as a function of matter density and dark energy density has been built, using the sets of data points obtained from the code using Python 2.7 in the later section. In this report, we also discuss the expansion histories of the universe given the energy density of components.

KEYWORDS: Cosmological expansion, Freidman Equations, curvature

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INTRODUCTION

One of the first theories explaining the dynamics of the universe was the Steady State Theory. The theory depicted as the universe having a steady state due to constant matter creation, hence having constant density. On the other hand, there was the evolutionary theory which stated that, density of the universe decreased over time as matter domination decreases. The universe is approximately homogenous and isotropic, and is now known to be in a state of accelerated expansion at the Hubble rate. Theoretically, a universe with the properties like ours expanding or contracting depending on certain cosmological factors like curvature k, scale factor a (t) and radius of curvature. Robertson-Walker metric describes the universe in this state of isotropy. We shall discuss the various expansion his Tories and shape of the universe depending on Friedman equations, which depend on the above characteristics. A graph plot built using Gnu plot, showing the age of the universe as a function of Ω_m and Ω_Λ has been built, using the sets of data points obtained from the code using Python 2.7 in the later section.

FREIDMAN EQUATIONS

Freidman developed a set of relativistic equations in the framework of general relativity to model the universe, and hence has been credited with developing the dynamics of an expanding universe. The Freidman equations do not include any particle interactions other than gravitational attraction, hence are solely built on the assumption of a 'pressure- less' universe.

These equations can also be derived using the Newtonian framework[1]. Assuming an observer to be moving in a uniformly expanding space of density ρ and the medium being isotropic, any point in the medium space can be taken as the center. A particle of mass m in the space can only be perturbed by the field in it's radius of curvature R_0. The scale factor measures the expansion rate of the universe and is a time-dependent quantity .If, between times tl and t2, the scale factor doubles in value, that tells us that the Universe has expanded in size by a factor two, consequently it will take twice as long to go from (x_1,y_1,z_1) to
(x_2,y_2,z_2) in the coordinate system (x,y,z).

$$V = \frac{GMm \; R_0}{-}$$

Total mass M is given by M = $4\pi\rho r^3/3$, which gives

$$V = - \frac{4Gmr^3\pi\rho \; 3}{}$$

The kinetic energy of the particle is $m\dot{r}^2/2$, where the velocity of the particle is $\dot{r}$. The total energy of a system is given by

$$U = T + V \qquad (1)$$

$$U = \frac{1}{2} mv^2 - \frac{4Gmr^2\pi\rho}{3} \qquad (2)$$

Substituting r by a(t)x, where r is the proper distance and a(t) is the scale factor and x is the comoving distance, we get:

$$U = \frac{1}{2} m(a\dot{x})^2 - \frac{4Gm(ax)^2\pi\rho}{3} \tag{3}$$

Multiplying each side by $2U/ma^2x^2$ *we get* :

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{4}$$

Equation (4) is the popular standard form of Freidmann equation. This can be modu- lated or transformed in a to obtain an acceleration equation, which can later be solved for various cases by changing the parameters.

THE ACCELERATION EQUATION

The acceleration equation describes the acceleration of the scale factor, an accelerating universe having pressure ’p’. It is notable that increase in pressure reduces the acceleration and vice versa[1]. By differentiating eq^n(4)

$$2\left(\frac{\dot{a}}{a}\right)\left(\frac{a\ddot{a}-\dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + 2kc^2\dot{a}/a^3 \tag{5}$$

The fluid equation gives the evolution of density with time (), and holds true only for adiabatic processes where dQ= 0 as there is no possible source for the heat to come from in Freidman universe. The fluid equation can be written as :

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) \tag{6}$$

Substituting in equation (5) we get:

$$\frac{\ddot{a}}{a} = -4\pi G \; -$$

$3 \quad \dot{\rho} + 3p/c^{2}\Sigma$ (7)

Equation (7) is the 'acceleration equation' which describes the dynamics of an accelerating universe.

EQUATION OF STATE PARAMETER *w*

The equation of state parameter, is a dimensionless constant which gives the relation between pressure and matter density. It can be expressed as:

$$w = p/\rho \tag{8}$$

It works on the assumption that there pressure at any point is associated or co-dependent on the matter density at that point. The w is similar to the equation of state in thermodynamics, taking the universe as a perfect isotropic fluid. In cosmology, the simplest assumption is p=0, which gives w=0 which corresponds to matter, so the pressure of non-relativistic matter (dust) is not enough to have any significant gravitational effect. The value of w for

radiation is w = -1/3 and dark energy Λ is w = -1

DENSITY IN RADIATION AND MATTER DOMINATED UNIVERSE

Next, we solve for density in Freidman equation for matter, radiation and Λ dominated universe. The curvature k can have 3 possible values k=-1,0,1 depending on whether it is an open, flat or closed universe respectively[2].

1. For matter dominated universe and k=0, p= 0

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0 \tag{9}$$

Since p=0,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho) = 0 \tag{10}$$

Multiplying by a^3 on both sides

which is the expanded form of

$$a^3\dot{\rho} + 3\dot{a}a^2\rho = 0 \tag{11}$$

$$\frac{d}{dt}(a^3\rho) = 0 \tag{12}$$

Therefore, it can be obtained that $a^3\rho$ =constant

2. For radiation dominated universe and k=0, p= $\rho/3$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \rho/3c^2\right) \tag{13}$$

Taking c = 1, the next equation is

$$a^4\dot{\rho} + 4\rho\dot{a}a^3 \tag{14}$$

Which can be written as $d(a^4\rho)/dt$, *hence* :

$$a^4\rho = constant \tag{15}$$

FREIDMAN EQUATION + ENERGY EQUATION + EQUATION OF STATE

For matter dominated model

$$a^3\rho = a^3\rho_0 \quad (16)$$

For radiation dominated model

$$a^4\rho = a^4\rho_0 \quad (17)$$

Taking $\left(\frac{\dot{a}}{a}\right) = H$, where H is the Hubble parameter which changes with time,

$$H^2 = \frac{8\pi G}{3}\left(\rho_r^0\left(\frac{a_0}{a}\right)^4 + \rho_m^0\left(\frac{a_0}{a}\right)^3 + \rho_\Lambda^0\right) - \frac{kc^2}{a^2} \quad (18)$$

This equation (18) is the standard mathematical expression which includes all three forms of energy matter, radiation and dark energy, which dominated the universe at differ- ent stages of time as well as the curvature constant k. [3]

The Critical density is given by

$$\rho_{cr} = \frac{3H^2}{8\pi G} \tag{19}$$

The next step is to find the dependence of scale factor a and H^{-1} for matter and radiation dominated universe for k=0 and plot a graph for $\log(H^{-1})$ and $\log(a)$.

For matter dominated universe, k = 0, p= 0, $a^3\rho$ = constant

Plugging the above parameters in equation number (18)

$$H^2 = \frac{8}{3}\pi G\rho_m \left(\frac{a_0}{a}\right)^3 \tag{20}$$

$$H = \sqrt{\frac{8}{3}\pi G\rho_m \left(\frac{a_0}{a}\right)^3} \tag{21}$$

$$\frac{da}{dt} = \sqrt{\frac{8}{3}\pi G\rho_m a_0^3}\,\frac{1}{a^{1/2}} \tag{22}$$

$$\int a^{1/2} da = \int \sqrt{\frac{8}{3}\pi G\rho_m a_0^3}\, dt \tag{23}$$

Let $a_0 = 1$, which is the current scale factor value

$$\frac{2}{3} a^{3/2} = \sqrt{\frac{8}{3}\pi G\rho_m a_0^3}\, t \tag{24}$$

Which gives,

$$\left(\frac{3}{\ }\right)^{2/3}$$

2/3

$$a = \quad {}_{2}H_{0} \qquad t \tag{25}$$

Also,

$$t = \frac{2}{} H^{-1} (26)\ 3$$

Therefore, the scale factor a is:

$$a = (H_0)^{2/3} (H^{-1})^{2/3} \tag{27}$$

Equation (27) gives the relation between a and H^{-1} for a matter dominated flat universe

For matter dominated universe, k = 0, p= $\rho/3$, $a^4\rho = constant$

$$H = \left(\frac{8}{3}\pi G\rho_m\left(\frac{a_0}{a}\right)^4\right)^{1/2} \quad (28)$$

$$\frac{da}{dt} = \left(\frac{8}{3}\pi G\rho_m a_0^3 \frac{1}{a}\right)^{1/2} \quad (29)$$

$$\int a\,da = \left(\frac{8}{3}\pi G\rho_m a_0^4\right)^{1/2}\int dt \quad (30)$$

Which results in,

$$a = (2H_0)^{1/2}t^{1/2} \quad (31)$$

$$a = (2H_0)^{1/2}\left(\frac{1}{2}H^{-1}\right)^{2/3} \quad (32)$$

For dark energy, k= 0, w = -1

$$\left(\frac{\dot{a}}{a}\right) = \left(\frac{8}{3}\pi G\rho_\Lambda\right)^{1/2} \quad (33)$$

$$\int \frac{da}{a} = \int \left(\frac{8}{3}\pi G\rho_\Lambda\right)^{1/2} dt \quad (34)$$

$$ln(a) = H_0 t \quad (35)$$

$$a = e^{Ht}, t = H^{-1} \quad (36)$$

Equation (36) gives the scale factor for a Λ dominated universe

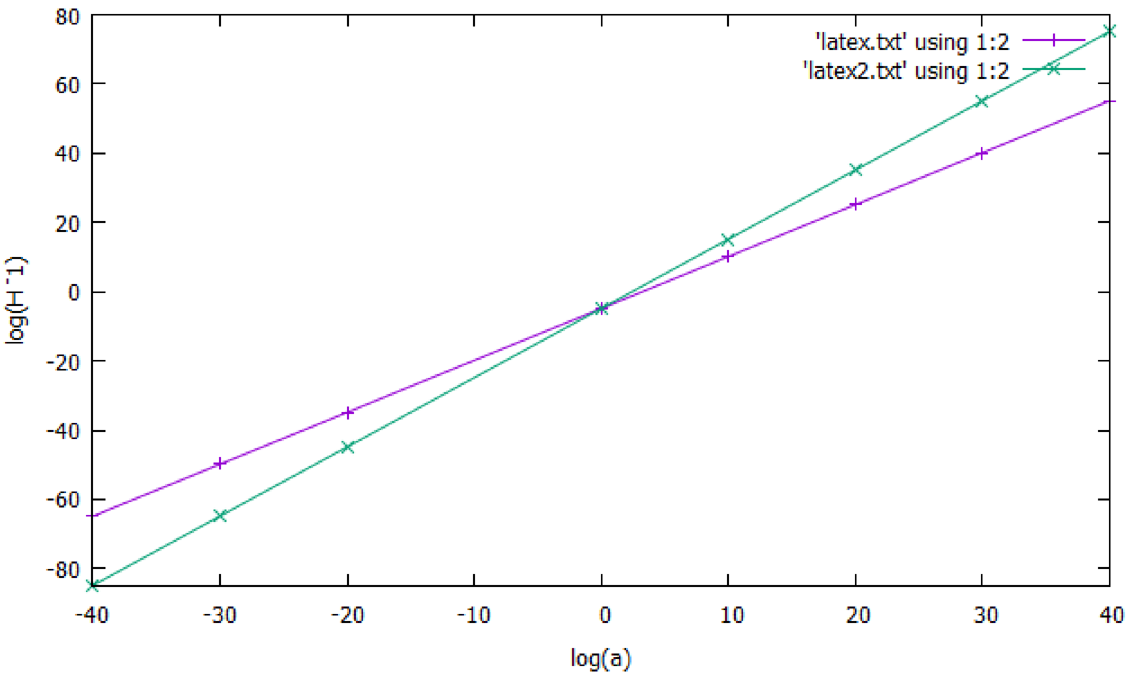
Below is the table giving the values of log(a) and log(H^{-1}) based on the solutions for matter dominated flat universe.

Table 1: **For MD**

log(a)	$\log(H^{-1})$
-40	-64.84
-30	-49.84
-20	-34.84
0	-4.84
10	10.155
20	25.55
30	40.155
40	55.155

Table 2: **For RD**

log(a)	$\log(H^{-1})$
-40	-84.84
-30	-64.84
-20	-44.84
0	-4.84
10	15.16
20	35.16
30	55.16
40	75.16



The violet line in the above graph represents for matter dominated era M.D $\propto a^3$ and the green line represents the radiation dominated era R.D $\propto a^4$. H^{-1} is called the Hubble time. Both the lines intersect at a point which is the point of transition from a radiation dominated era to a matter dominated one at the value of log(a) and log (t) corresponding to that point.

AGE OF THE UNIVERSE AS A FUNCTION OF COSMOLOGY- CAL PARAMETERS

We first obtain a general solution of the Freidman equation from (18)

$$H^2 = \frac{8\pi G}{3}\left[\rho_r^0\left(\frac{a_0}{a}\right)^4 + \rho_m^0\left(\frac{a_0}{a}\right)^3 + \rho_\Lambda^0\right] - \frac{kc^2}{a^2} \quad (37)$$

Multiplying and dividing (37) by H_0^2

$$\left(\frac{H}{H_0}\right)^2 = \frac{8\pi G}{3H_0^2}\left[\rho_r^0\left(\frac{a_0}{a}\right)^4 + \rho_m^0\left(\frac{a_0}{a}\right)^3 + \rho_\Lambda^0\right] - \frac{kc^2}{a^2H_0^2}$$

Now, $H_0 \; = \; 3H^2 \; \rho_r \; a \; \rho \; + \rho_m \; 3H^2 \; a \; + \rho_\Lambda$

$$- \frac{}{a^2H^2} \qquad (38)$$

$$_{cr} = \frac{}{8\pi G} \qquad (39)$$

$$\left(\frac{H}{H_0}\right)^2 = \frac{\rho_r^0}{\rho_c}\left(\frac{a_0}{a}\right)^4 + \frac{\rho_m^0}{\rho_c}\left(\frac{a_0}{a}\right)^3 + \frac{\rho_\Lambda^0}{\rho_c} - \frac{kc^2}{a^2 H_0^2} \qquad (40)$$

$$\Omega_m = \frac{\rho_m}{\rho_c}, \Omega_r = \frac{\rho_r}{\rho_c}, \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} \qquad (41)$$

Substituting (41) in (38), we get

$$\left(\frac{H}{H_0}\right)^2 = \Omega_r\left(\frac{a_0}{a}\right)^4 + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + \Omega_k\left(\frac{a_0}{a}\right)^2 \quad (42)$$

Ω_k is the curvature parameter which is equal to $-k/a_0^2H_0^2$, taking a_0^2, and another variable x, where $x = a/a_0$ and multiplying x^2 on both sides:

$$\left(\frac{H}{H_0}\right)^2 x^2 = \Omega_r x^{-2} + \Omega_m x^{-1} + \Omega_\Lambda x^2 + \Omega_k \quad (43)$$

Now, H = $\left(\frac{\dot{a}}{a}\right)$, $\dot{a} = da/dt, x = a/a_0$ so, $\left(\frac{da}{d}\right) = \left(\frac{dx}{d}\right)$

$$L.H.S = (dq/dt)^2/H_0^2 \quad (44)$$

$$\int dt = \int \frac{x(t)}{H_0^2\sqrt{\Omega_r^0 x^{-2} + \Omega_m^0 x^{-2} + \Omega_\Lambda^0 x^2 + \Omega_k}} \quad (45)$$

Equation (45) gives the age of the universe t which is a dimensionless function of matter density parameter Ω_m ,radiation density parameter Ω_r and DE density parameter Ω_Λ , and Ω_k can be written in terms of other parameters as below:

$$\Omega_k = 1 - \Omega_m - \Omega_r - \Omega_\Lambda \quad (46)$$

The FRW cosmology is completely based on these parameters, and the objective of observational cosmology is to find the values of these parameters. For our universe Ω_r =
7 10^{-5} which is really a very negligible value to affect the time t, while Ω_m = 0.3 and Ω_r = 0.7. For open universe with negative curvature Ω < 1, for a flat universe Ω = 1 and corresponding to closed universe with positive curvature Ω > 1.

DYNAMIC AND EXPANSION HISTORIES OF THE UNIVERSE FROM FREIDMAN MODELS

Given the total energy content of a Universe, given as a sum of different perfect fluids such as radiation, pressure less matter, a cosmological constant etc, it is not obvious what kind of expansion histories are possible, e.g. if the model has a Big Bang, if it will expand forever, if it accelerates etc. We derive a rescaled Freidman equation, which is of the form of energy conservation equation in classical dynamics containing K. E and P. E terms. Rewriting equation (43):

$$\left(\frac{H}{H_0}\right)^2 x^2 = \Omega_{r0}x^{-2} + \Omega_{m0}x^{-1} + \Omega_{\Lambda 0}x^2 + \Omega_{k0} \tag{47}$$

τ is a dimensionless measure of time defined as $\tau = tH_0$, H_0 being the Hubble constant. Multiplying 1/2 on both sides:

$$\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 = \frac{1}{2} \times \left(\Omega_{r0}x^{-2} + \Omega_{m0}x^{-1} + \Omega_{\Lambda 0}x^2 + \Omega_{k0}\right) \tag{48}$$

$$\frac{1}{2}(\dot{x})^2 + U_{eff} = \frac{\Omega_k}{2} \qquad (49)$$

Where,

$$U_{eff} = -\frac{1}{2}\left[\Omega^0_\Lambda x^2 + \frac{\Omega^0_m}{x} + \frac{\Omega^0_r}{x^2}\right] \qquad (50)$$

$U_{eff}(x)$ is the effective potential energy in cosmology, similar to potential energy in classical dynamics. Equation (49) is the energy equation K + U = E for a particle moving one dimensionally along coordinate a with kinetic energy $\frac{1}{2}(\dot{x})^2$, potential energy $U_{eff}(x)$ and total energy $\Omega_k/2$. To understand the expansion histories we plot the graph for $U_{eff}(x)$ v/s x for different cosmological parameters : matter, radiation, Λ and curvature. We plot a graph corresponding to each parameter based on the values of x.

For x 1

If x 1, the radiation term becomes the most dominant as $-\frac{\Omega_r}{x^2}$, while the matter and DE density terms are comparatively negligible while where x = a/a_0, then:

$$U_{eff}(x) \approx -\frac{\Omega^{(0)}_r}{x^2} \qquad (51)$$

The equation is of the form y = -c/x^2 hence the graph obtained is of the same form.

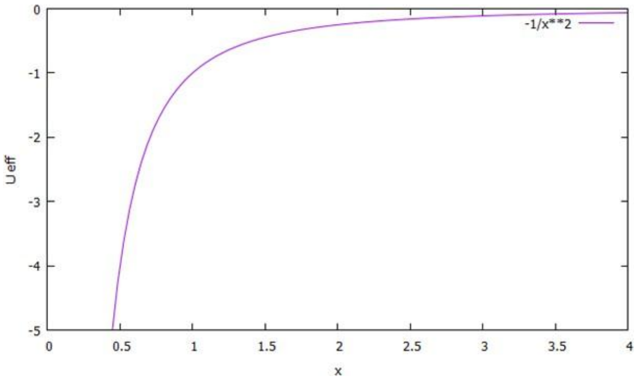


Figure 1: The graph for effective potential v/s scale factor for the above case

For x 1

If x1, the dark energy term becomes the most dominant as $\Omega_\Lambda x^2$, while the matter and DE density terms are comparatively negligible while where x = a/a_0, then:

$$U_{eff}(x) \approx \Omega_\Lambda x^2 \quad (52)$$

The equation is of the form y = $-cx^2$ hence the graph obtained is of the same form. This corresponds to the current accelerated phase when the cosmological constant is dominating the energy content of the Universe. The scale factor then grows indefinitely while the matter density approaches zero

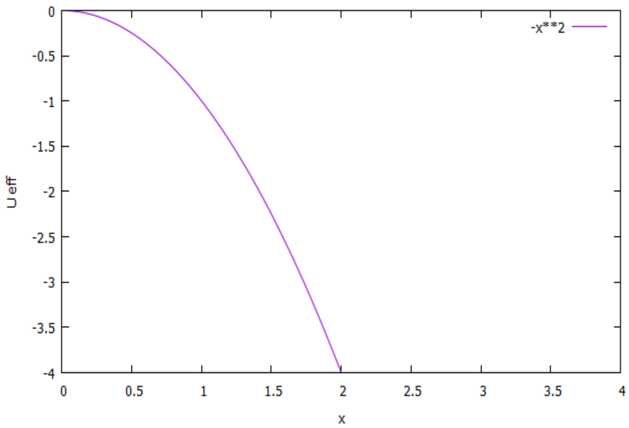


Figure 2: The graph for effective potential v/s scale factor for scale factor much greater than 1

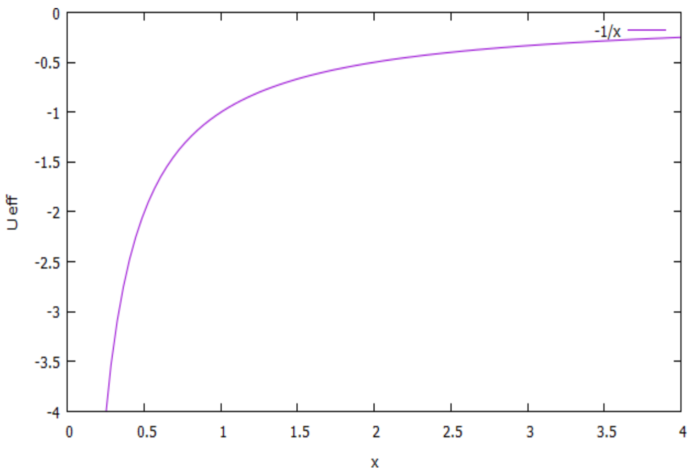
For matter dominated model

Another way to plot the dependence of the effective potential energy on x = a/a_0 for a matter dominated model with zero curvature, k = 0 is taking:

$$U_{eff}(x) \approx \frac{\Omega_m}{x} \qquad (53$$

The equation is of the form -1/x, so the graph obtained is as below:

Figure 3: Effective potential v/s scale factor for MD model



The above graph depicts that for a matter dominated model, the scale factor keeps on increasing, and U_{eff} becomes less and less negative. Due to this, it can be concluded that pressure less matter decreases the rate of expansion.

For curvature k = -1 and k = +1 : : Open and closed universe

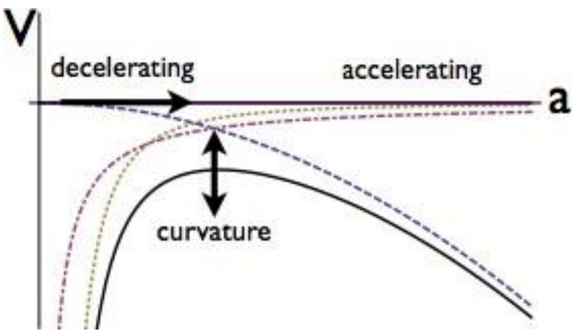
The curvature parameter Ω_k is of the form of total energy in equation (49). For an open universe with k = -1, Ω_k *becomes positive, while fork* = +1*closeduniverse*Ω_k stays negative.

Ω $= - \frac{k}{}$ $E_k =$ (54)

$2a^2H^2$

0 0

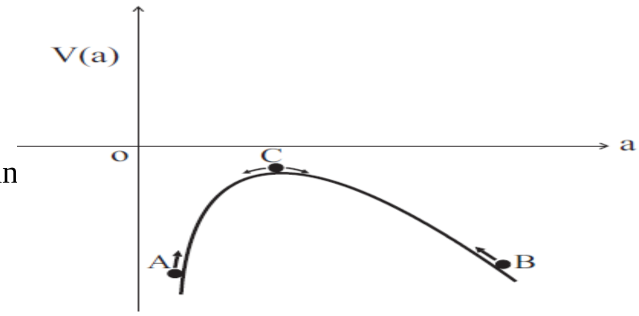
Figure 4: **Graphs of V v/s a(t) for different cases** [4]



The above graph shows what the the plot for U_{eff} v/s x looks like for a universe with positive or negative curvature. The curvature parameter relates to the others as in Ω_k = 1 Ω_0, where $\Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda$. It can be derived that the system undergoes an accelerated expansion in the beginning as the scale factor increases, the potential increases and reaches

a certain maximum called the ”potential barrier” and undergoes decceleration later on and the scale factor decreases.

Figure 5: **Different configurations of the universe as a particle**



- FIG.[4]: Plot of the effective potential V (a) versus the scale factor a. For configuration (A) the motion of the sys- tem begin (B) the motion of system be- ginning from a = 0 and a = 1 respectively. (C) corresponds to quasi-static solution, which becomes unstable under mi- nor fluctuations.

From equation (54) if:

$$k = 0, E = 0; k = 1,$$
$$E < 0; k = -1, E > 0 \qquad (55)$$

The negative total energy E ł0 lies below the abscissa, while E 0 lies above the abscissa, whereas E = 0 lies on the x-axis itself. Supposing the universe a particle rolling up and down the slopes. For region A the motion of the system begins from x=0, reaches a certain peak value of x. So the scale factor keeps on increasing, P.E increases while the kinetic energy decreases. The K.E at the top is insufficient to cross the barrier, hence the system bounces back towards the decreasing scale factor and increasing negative potential U_{eff}.

Region (B) : The system starts its motion from a = +∞ and moves towards decreasing scale factor and increasing P.E(a). The system reaches a potential barrier which is not possible to overcome due to insufficient K.E, so the system rolls down the slope with increasing scale factor and decreasing P.E/ U_{eff}, hence the system continues to expand forever, which similar to a case of Λ dominated universe. For a closed universe, the total energy goes E ¡ 0

where the universe expands in the beginning, the U_{eff} and x reach a certain peak value and rolls back to a = 0, which closely depicts the region (A). As for an open universe with k =-1, there is no energy barrier for the system since the potential barrier for E ¿ 0 lies above the abscissa , hence it undergoes eternal inflation.

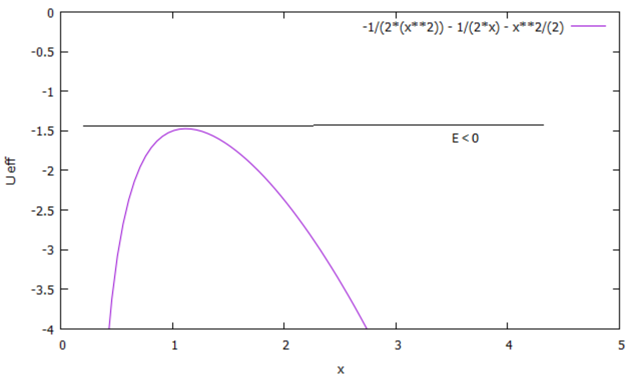


Figure 6: The graph for effective potential v/s scale factor for the above case

The above image is the graphical representation of U_{eff} v/s x for an open universe E 0, k = -1. The effective potential is of the form:

$$U_{eff} = -\frac{1}{2}\left[\frac{1}{x^2} + \frac{1}{x} + \right.$$

$\frac{1}{} \quad x^2 \Sigma$ (56)

$U_{eff} \approx -\frac{1}{2} - \frac{1}{x^2} \Sigma$; for x 1

$U_{eff} \approx 3$; for x=1 (57)

$U_{eff} \approx x^2$; for x 1

The earlier portion of the graph starting from x = 0 to x $\approx$ 1.2 is the plot for the first equation in (57), the peak aft is the inverse graphical representation of the second equation in (57), while the downward slope gives the inverse plot for the third equation due to the (-) sign in (56). For an open universe, at x(t) $\approx$ 1.2 the universe expands eternally and the model does not have a Big Bang. At x(t) 1.2, the universe has a constant negative potential and is in a stable state. The region from 0 a 0.8 , shows a region transitioning/ beginning from a Big Bang at x = a/a_0 = 0

Freidmann acceleration equation :

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + \frac{3p}{c^2}\right) \tag{58}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + \frac{3\omega\rho}{c^2}\right) \tag{59}$$

$$= -\frac{4}{3}\pi G\rho\,(1+3\omega)$$

In classical dynamics F = m = -dV/dx. Therefore:

$$\ddot{a} = -\frac{dV}{da} \tag{60}$$

Which means the acceleration of the universe depends on ω from (59) and V from (60)

$\omega > -1/3$; $\omega < -1/3$; $\omega = -1/3$;

$$\frac{\ddot{a}}{a} < 0 \tag{61}$$

$$\frac{\ddot{a}}{a} > 0 \tag{62}$$

$$\frac{\ddot{a}}{a} = 0 \tag{63}$$

Since acceleration $\left(\frac{\ddot{a}}{a}\right)$ is dependent on potential as $\left(\frac{dV}{dx}\right)$, when $\left(\frac{\ddot{a}}{a}\right)$ is greater than 0, $\left(\frac{dV}{dx}\right)$ should be negative, so $\omega < -1/3$. Similarly for $\left(\frac{\ddot{a}}{a}\right) < 0$, the term $\left(\frac{dV}{dx}\right)$ should be positive, so $\omega > -1/3$. If $\left(\frac{\ddot{a}}{a}\right) = 0$, $\left(\frac{dV}{dx}\right)$ should be zero. If $\omega > -1/3$, as would be the case with only matter and radiation With dark energy there is a new twist: since the dark energy density decreases more slowly than that of matter or radiation, as the Universe ex- pands dark energy eventually dominates the third term in U_{eff}. Thereafter, U_{eff} decreases monotonically, since ω_{DE} < -1/3, approaching -∞ as a tends to ∞. Within GR, accelerated expansion cannot be explained by any known form of matter or energy but can be accom-
modated by a nearly smooth form of energy with large negative pressure, known as dark energy, that accounts for about 75 % of the Universe.

9 SOLUTIONS OF SCALE FACTOR a(t) AND TIME t FROM FREI- DMANN EQUATIONS FOR M.D AND R.D MODELS WITH CUR- VATURE

The relative expansion of the universe is parameterized by a time dependent scale factor.

We solve the Freidmann equations for different cases:

For matter dominated, k = 1 closed universe

$(\dot{a}a) = -\pi G\rho$ $\quad \frac{8}{3}$ $\quad \rho_0 \left(\frac{a_0}{a}\right)^3 - \frac{1}{a^2}$ (64)

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3a}\rho_0 a_0^3 - 1 \quad (65)$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi G}{3a}\rho_0 a_0^3 - 1} \quad (66)$$

Taking $a_0 = 1$ $\int dt = \int \frac{da}{\sqrt{\frac{8\pi G}{3a}\rho_0 - 1}}$ (67)

Conformal time is the time it would take light to travel from the observer's point to the farthest observable distance. It is denoted by η, and follows the below relationship:

$$d\eta = \frac{dt}{a} \quad (68)$$

Therefore,

$$\int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G}{3a}\rho_0 - a^2}} \quad (69)$$

Let $\left(\frac{4}{3}\pi G\rho_0\right) = A$

Now, $\int d\eta =$

$$\frac{da}{\sqrt{2Aa-a^2}} \qquad (70)$$

$$1-\left(\frac{a-A}{A}\right)^2 = \frac{2aA-a^2}{A^2} \qquad (71)$$

$$\int_0^a \frac{da}{A\sqrt{1-\left(\frac{a-A}{A}\right)^2}} \qquad (72)$$

Let a - A/ A = x, therefore dx = da/A. Let x = sinθ , dx = cos$\theta d\theta$ So θ = sin^{-1}x $\int_0^x \frac{\cos\theta d\theta}{\cos\theta}$ (73)

$$\int_0^x d\theta \qquad (74)$$

Therefore, putting the limits , x = sonθ

$$[sin^{-1}x + c]_0^x \qquad (75)$$

Putting $x = \left(\frac{a-A}{a}\right)$

$$\int d\eta = \sin^{-1}\left(\frac{a-A}{A}\right) + \frac{\pi}{2} \qquad (76)$$

$$\sin\left(\eta - \frac{\pi}{2}\right) = \left(\frac{a-A}{A}\right) \qquad (77)$$

$$-\cos\eta = \left(\frac{a-A}{A}\right) \qquad (78)$$

Therefore, the solution for scale factor is :

$$a = A(1-\cos\eta) \qquad (79)$$

Now, $A = \left(\frac{4\pi G}{3}\rho_0\right) = H_0^2$q, where q is the decceleration parameter

$$H^2 = \frac{1}{2q-1} \qquad (80)$$

$$A = \frac{q_0}{2q_0-1}$$

For matter dominated universe, q = $\Omega_0/2$; A = $\left(\frac{\Omega_0}{2(\Omega_0-1)}\right)$. Hence, the solution for a is:

$$a = \left(\frac{\Omega_0}{2(\Omega_0-1)}\right)(1-\cos\eta) \qquad (81)$$

Now, dτ = $a d\eta$, therefore:

$$t - t_0 = \int a d\eta \qquad (82)$$

Let t_0 at η = 0 which is the case for Big Bang,

$$t = A(\eta - \eta\sin\eta) \qquad (83)$$

Thus the solutions for time dependent scale factor a(t) and t for matter dominated closed universe are

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\eta) \quad (84)$$

$$t = \frac{\Omega_0}{2(\Omega_0 - 1)}(\eta - \sin\eta)$$

For radiation dominated, k = 1 closed universe, Ω

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8}{3}\pi G\rho_0\left(\frac{a_0}{a}\right)^4 - \frac{k}{a^2} \quad (85)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3a^4} - \frac{1}{a^2} \quad (86)$$

$$\frac{da}{dt} = \frac{8\pi G\rho_0}{3a^4} - \frac{1}{a^2} \quad (87)$$

Since, $d\tau = a d\eta$:

$$\int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0}{3a^2} - 1}} \quad (88)$$

$$\int d\eta = \int \frac{da}{\sqrt{8\pi G\rho_0 - a^2}} \quad (89)$$

Let $\left(\frac{8\pi G\rho_0}{3}\right) = A$

$$\int \eta = \int \frac{da}{\sqrt{A - a^2}} \quad (90)$$

$$\int d\eta = \int \frac{da}{\sqrt{A\left(1 - \left(\frac{a}{\sqrt{A}}\right)^2\right)}} \approx \int \frac{dx}{\sqrt{1 - x^2}} \quad (91)$$

where $x = \frac{a}{\sqrt{A}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c \quad (92)$$

Therefore,

$$\int \frac{da}{\sqrt{\;}}$$

-1 $\cdot$ $\underline{a}$ Σ —

On integrating dη $\cdot$ Σ_2 $\sqrt{a}$) $= sin$ $\sqrt{A}$ (93)

$A(1-$

A

$\sqrt{A}sin\eta = a$ — (94)

$\int ad\eta = \int dt$

(95)

$t - t_0 = \int \sqrt{A}sin\eta$

Let η = 0, so t= 0, which makes $t_0 = \sqrt{A}$, because $t - t_0 = \sqrt{A}cos(0)$. Rewriting the above integral:

$t = \sqrt{A}cos(\eta) + \sqrt{A}$ — (96)

$t = \sqrt{1 - cos\eta}$ ———— (97)

$A = 2H_0^2 q; q = \frac{4\pi G\rho}{3H_0^2}$ (98)

$$A = \frac{2q}{2q-1} \tag{99}$$

Therefore,

$$a = \frac{2q}{2q-1} sin\eta \tag{100}$$

$$t = \frac{2q}{2q-1}(1 - cos\eta)$$

For matter dominated, k = -1 open universe, $\Omega > 1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 + \frac{1}{a^2} \tag{101}$$

$$\frac{da}{dt}^2 = \frac{8\pi G\rho}{3a} + 1 \tag{102}$$

On simplifying, we get:

Since we now that dη =dt/a

Let A = $4\pi G\rho$ 3 –

$$\int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho}{3} + 1}} \quad (103)$$

$$\int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho a}{3} + a^2}} \quad (104)$$

$$\eta - \eta_0 = \int_0^a \frac{da}{\sqrt{2aA + a^2}} \quad (105)$$

For the above equation, $A\left(\frac{A+a}{A}\right)^2 - 1 = 2Aa + a^2$, which can be replaced in the denominator. The equation becomes:

$\int \sqrt{\ 1}$

$\frac{}{} \ x^2 \quad 1$ $\qquad$ $-$ $\qquad$ (106)

$\frac{}{} \quad = ln|\sqrt{x^2 - 1} + x| + c$

Where x = $\left(\frac{A+a}{A}\right)$. Equation (105) becomes:

$\frac{A + a}{}$

$\left(\frac{A + a}{}\right)^2$

$= ln$ $\qquad$ $A \quad - 1 + \quad A$ $\qquad$ (107)

$\frac{a}{}$

$= ln$ $\qquad$ $\left(a \left(\frac{2aA + a}{}\right)^2 + 1 + \right)$ (108)

A

$\frac{}{} A$

$\sqrt{}$

The identity $cosh^{-1}x = ln(x + \sqrt{x^2 - 1})$ applies here:

$\eta = cosh^{-1} \left(1 + \frac{a}{A}\right)$ $\qquad$ (109)

Therefore, the solutions for a(t) and t are as given below:

$$cosh\eta = \frac{a + A}{A} \qquad (110)$$

Since dt = $\int d\eta a$

$$a = A(cosh\eta - 1)$$

$$t = A(sinh\eta - \eta) \qquad (111)$$

For radiation dominated case, k = -1 open universe

From the Freidmann equation, $\left(\frac{\dot{a}}{a}\right)^2 - \quad = \frac{}{3}\pi G\rho_0$

$$\left(\frac{a_0}{a}\right)^4 \quad \frac{1}{+a^8} \tag{112}$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi G\rho}{3a^2}+1} \tag{113}$$

$$\int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho}{3a^2}+1}} \tag{114}$$

$$\int_0^a d\eta = \int \frac{da}{\sqrt{A+a^2}} \tag{115}$$

$$\int_0^a \frac{d\eta}{\sqrt{}} = \int \frac{da}{\sqrt{A}\sqrt{A+(a/\sqrt{A})^2}} \tag{116}$$

where, $A = \frac{8\pi G\rho}{3}$. Let $a/\sqrt{A}$ be x. So, $dx = da/\sqrt{A}$

$$= \ln|\sqrt{1+x^2}+x| + c$$

$$= \sinh^{-1}x \tag{117}$$

$$= \sinh^{-1}\left(\frac{a}{\sqrt{A}}\right)$$

Again, the requirement $\eta = 0$ at a = 0 sets $\eta_0 = 0$, so we have the solution for a : $a = \sqrt{A}sinh\eta$ (118)

Therefore, since A = .1 2q0

$-2q_0$

2q

$a =$ $\frac{0}{}$ $sinh\eta$ (119)

$1 - 2q_0$

Time $\int dt = \int ad\eta$, so integrating the above equation we get,

$t - t_0 = \sqrt{A}cosh\eta$ (120)

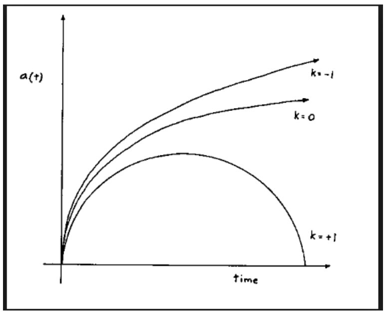
At Big Bang $\eta = 0$, $hence t = 0$, $so t_0 = \sqrt{A}$,

$t =$ 2q0 ($cosh\eta$ 1) (121)

$1 - 2q_0$

Equation (119) and (121) are the solutions for a(t) and t for a radiation dominated open universe.

Figure 7: The image below is the graphical representation of open closed and flat universe as a function of scale factor a(t) and time t [5]



In absence of a cosmological constant, the fate of the universe depends on the value of k. In this case, the fate of the universe is actually analogous to firing a rocket on Earth. If the velocity of the rocket is too low, it will be less than the escape velocity. Its trajectory will be a parabola, and it will crash back to the ground. If its velocity is equal to the escape velocity, it will just barely escape, and might go into orbit. If its velocity is higher, it will leave Earth completely, and head off into interplanetary space. You could think of each of these as analogous to different values of k. If k = +1, the universe will not be expanding fast enough to overcome its own gravity. The expansion will slow down, stop, and reverse, and the universe will recollapse and end in a Big Crunch 5. If k = 0, the universe will be just fast enough for it to escape this fate. If k = -1, the velocity will be higher than that, and it will also expand indefinitely. Therefore, a universe with positive curvature will exist for a finite length of time, while a universe with flat or negative curvature will exist for an infinite length of time. Therefore, universes with k = +1 are both spatially and temporally closed, while universes with k = 0 or k = -1, are both spatially and temporally open.

CALCULATING THE AGE OF THE UNIVERSE FOR $\Omega_m + \Omega_\Lambda = 1$, where $\Omega_k = 1$

We know that $\Omega_m + \Omega_\Lambda + \Omega_r + \Omega_k = 1$ and $\Omega_0 = \Omega_m + \Omega_\Lambda + \Omega_r$. We consider a universe dominated by matter and Dark energy, which is the current scenario and $\Omega_r \approx 5 \times 10^{-7}$ which is a negligible value . Equation (45) gives the age of the universe as derived previously, where x = a/a$_0$

$$\int_0^{t_0} dt = t_0 \quad (122)$$

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{dx}{\sqrt{(1-\Omega_0)+\Omega_r x^{-2}+\Omega_m x^{-1}+\Omega_\Lambda x^2}} \quad (123)$$

For Ω_m, Ω_Λ universe $\Omega_m + \Omega_\Lambda = 1 = \Omega_0 = 1$

$\Omega_\Lambda = 1 - \Omega_m$; $\Omega_0 = 1$ and Ω_r is negligible

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{dx}{\sqrt{0+0+\Omega_m x^{-1}+(1-\Omega_m)x^2}} \quad (124)$$

$$= \frac{1}{H_0} \int_0^1 \frac{dx}{\sqrt{\Omega_m}\sqrt{x^{-1}}\sqrt{1+\left(\frac{1-\Omega_m}{\Omega_m}\right)x^3}} \quad (125)$$

$$= \frac{1}{H_0} \int_0^1 \frac{dx\sqrt{x}}{\sqrt{\Omega_m}\sqrt{1+\left(\frac{1-\Omega_m}{\Omega_m}\right)x^3}} \quad (126)$$

Let $x^3 = \frac{\Omega_m}{1-\Omega_m} sinh^2\varphi/2$, so that we get an equation of the form:

$$\frac{1}{\sqrt{1+sinh^2\varphi/2}} \quad (127)$$

On substituting x^3 in (126) $\frac{1}{} \int^{1-} \frac{dx\sqrt{x}}{}$

(128)

$= H_0 \quad {}_0\sqrt{\Omega_m} \quad \sqrt{1 + sinh^2\varphi/2}$

Differentiating x^3 we get,

$dx = \frac{1}{3x^2} \left(\frac{\Omega_m}{1-\Omega_m} \right) sinh\frac{\varphi}{2} cosh\frac{\varphi}{2}$ (129)

Substituting it in equation (128)

$$= H_0 \quad 0 \quad \frac{3x \quad \sqrt{\Omega_m} \quad 2 \quad 2}{\Omega_m cosh\varphi/2} \tag{130}$$

$$\frac{1}{} \int_1 \sqrt{x}\ \frac{1}{2}\ \frac{\Omega_m}{1} \Sigma\ sinh^{\varphi}\ cosh^{\varphi}\ d\varphi \qquad \frac{1}{} \int_1 \sqrt{x} \sqrt{\Omega_m} sinh^{\varphi}\ d\varphi$$

$$H_0\, 0 \qquad \frac{3x^2}{(1-\Omega_m)} \quad 2 \tag{131}$$

It can be obtained that::

$$sinh\frac{\varphi}{2} = \frac{1-\Omega_m}{\Omega_m} \tag{132}$$

Putting this value in (131) current time t_0 will give:

$$t = H^{-1}_0 \int_0^1 \frac{d\varphi}{\sqrt{1-\Omega_m}} \quad 0 \tag{133}$$

$$t_0 = H_0^{-1} \frac{(\varphi_1 - \varphi_0)}{3\sqrt{1-\Omega_m}} \tag{134}$$

If x = 0 in $x^3 = \frac{\Omega_m}{1-\Omega} \Sigma\ sinh^2\varphi/2; \varphi = 0$, therefore $\varphi = 0$

If x = 1 in $x^3 = \frac{\Omega_m}{1-\Omega} \Sigma$, $therefore$

$$\varphi = 2sinh^{-1} \frac{1-\Omega_m}{\Omega_m} \tag{135}$$

Since, 1 - $\Omega_m = \Omega_\Lambda$

(136)

$$\frac{1}{t_0} = \frac{2}{H_0\,3\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\right)$$

Equation (136) is the age of the universe for a radiation and matter dominated flat universe with $\Omega_k = 1$. To verify this, let $\Omega_\lambda = 0.7, \Omega_m = 0.3$, which are the current values of density parameters for dark energy and matter respectively:

$$t_0 = \frac{1}{H_0}\left(0.8 \sinh^{-1}(1.52)\right) \qquad (137)$$

On solving, we get

$$t_0 = 0.96 H_0^{-1} \qquad (138)$$

Equation (138) closely gives the current value of the age of our universe which is matter and Ω dominated flat universe. Since this calculation involved two of the four parameters, the calculation has been simpler and can be done analytically as shown above. We try taking a few different values for Ω_m and Ω_Λ to obtain t_0 for the same.

1. Let $\Omega_m = 0.6$; $\Omega_\Lambda = 0.4$

Table 3: **Age of the universe for different parameters**

(144)

Ω_m	Ω_Λ	t_0
0.3	0.7	$0.96H_0^{-1}$
0.6	0.4	$0.8H_0^{-1}$
0.9	0.1	$0.689H_0^{-1}$
0.1	0.9	$1.27H_0^{-1}$

The term H_0^{-1} is the t_h, which is the Hubble time, where as usual we used H0 = 72 km $s^{-1} Mpc^{-1}$. $t_H = H_0^{-1}$ = 13.6 Gyr, where 1 Gyr = 10^9 yrs. Finding the estimated age of the universe through practical experiments and verifying them with the age theoretically implied by the cosmological models, is a valid trial to check these models. Cosmological constants have negative pressure which is why they are responsible for expansion of the universe. For Big Bang models with zero cosmological constant and positive matter density, the actual age was found to be considerably younger than this Hubble time, depending on the density of matter. After the The presence of dark energy implies that the universe was expanding more slowly at around half its present age than today, which makes the universe older for a given value of the Hubble constant.

FOR A CLOSED OR OPEN UNIVERSE WITH Ω>1 or Ω<1, (1 - Ω_k) f=0

$$\int_0^{t_0} dt = t_0 \tag{145}$$

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dx}{\sqrt{(1-\Omega_0)+\Omega_r x^{-2}+\Omega_m x^{-1}+\Omega_\Lambda x^2}} \tag{146}$$

It is a tedious task to solve the above integral analytically. So, taking the numerical approach I devised a code in Python(Spyder 2.7) to solve for the value of t_0 for different t_0 for different ranges of scale factor a(t).

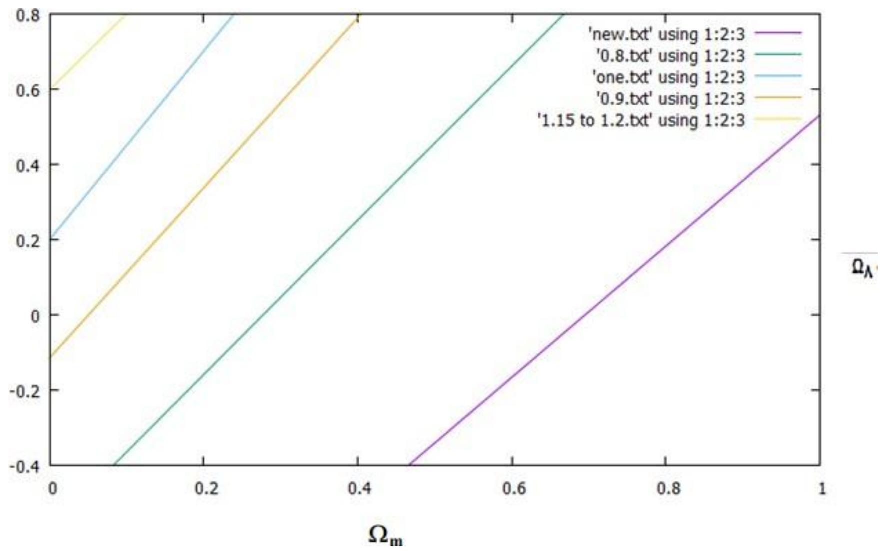
CODE:

```
From scipy. integrate import  quadimportnumpy def    integrand ( a ) :
return 1 /   (((omega r) _/ (a / a0 )** 2 +
i    / (a / a0 ) + j * (a / a0 ) ** 2 + (1 − omega o)) _ ** 0.5 )
omega r = 0 . 0
omega o = 0 _ 0a0 = 1
x = numpy . arange (0.0 ,  1.01 ,  0.1)y = numpy . arange ( −1.0 ,1.01 ,0.1)

list  =[]
for  i  in x:
for     j  in y:
ans ,  err  =  quad (integrand ,  0,  1)
omega o  =  omega r  +  j  +  i if    0.95 < ans < 1.05:
print(list)    _         _
list . append ( ( i , j , ans ) )
```

The code given in the previous page is specifically made for Python as mentioned pre- viously. The answer, which is the age of the universe can be obtained in various ranges and these points belonging to a certain range can be represented on the graph using contour lines, each coloured line belonging to a specific range for the age t_0. It is implicit from the below graph that corresponding to different values of matter and DE energy the age of the universe ,as a multiple of Hubble time t_H , varies with values of Ω_m and Ω_Λ.

Figure 8: **The image in the side is the graphical representation of the age of the universe as a function of Ω_m and Ω_Λ**



Another manifestations of these calculations and graph is that with the increase in age of the universe, the matter density decreases and on the contrary the Dark energy density increases or vice versa. The current age of the universe is around 13.8 billion years. For matter dominated universe, a(t) $\propto t^{2/3}$; increase in matter causes increase in deceleration (as an effect of gravity) as a result. As acceleration decreases (due to extra matter) a(t) decreases which in turn causes decrease in age/time. Similarly, as matter density decreases the acceleration increases, hence the scale factor also increases. The age of the universe increases with decrease in matter density.

As Ω_Λ increases, the acceleration of the universe increases and hence the scale factor exponentially increases since a(t) $\propto e^{Ht}$. As the time-dependent scale factor increases the age of the universe also increases. For an extreme case such as $\Omega_\Lambda = 1$, the age of the universe is infinite.

General relativity predicts a 'Big Bang' singularity, the beginning of the universe as a singular state of infinite density which is then followed by expansion with various stages[6]. Not diving into specifics, the earliest dominating energy was radiation, which diminished the effects of the spatial curvature term Ω_k and the universe must be flat then have been flat then. As the radiation dominated era wipes off and matter becomes pre-dominant the curvature term emerges due to nature of the matter The positive curvature models recollapse and end in a 'Big crunch'.

1. Liddle, A ."*An introduction to modern cosmology* ".2nd Edition, 1993; 32-40.
2. Dr. Balša Terzić . "*Lecture 5 : Solutions of Freidmann Equations"* [online]. 2018 [cited 17 Jul. 2018] Available from URL: http://nicadd.niu.edu/~bterzic/PHYS652/Syllabus.pdf
3. Hartle, J. "*Gravity"*. San Francisco: Addison-Wesley, 2003; 380-393
4. Mörtsell, E. "*Cosmological histories from the Friedmann equation: the Universe as a particle*." European Journal of Physics *2016*; 37(5) :6-8.
5. J. A. Fireman et al. " Dark Energy and the Accelerating Universe*"* [online]. 2018 [cited 20 Jul. 2018]. Available at URL: https://ned.ipac.caltech.edu/level5/March08/Frieman/Frieman9.html
6. Chavanis, P. " A cosmological model describing the early inflation, the intermediate decelerating expansion, and the late accelerating expansion by a quadratic equation of state" [online] Arxiv.org;. 2018 [cited 12 Jul. 2018]. Available at URL: http://arxiv.org/abs/arXiv:1309.5784