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Solution of Fractional Differential Equations By Adomian Decomposition Method With Chebyshev Polynomials

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ABSTRACT: We study the nonlinear fractional differential equations using Adomian Decomposition with Chebyshev polynomials. The source terms are represented in terms of Taylor series and Chebyshev basis elements. The schemes are tested for some examples and the validity of the results are compared with the exact solution obtained by taking the value of fractional derivative α as integer.

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1. INTRODUCTION:

George Adomian[1, 2, 3, 4] introduced a powerful method named as Adomian decomposition method (ADM) for solving linear and nonlinear functional equations. Adomian decomposition method, is a well-known for analytical approximate solutions of linear or nonlinear differential equations. In the last two decades, extensive work has been done using ADM, as it provides analytical approximate solutions for nonlinear equations and considerable interest in solving fractional differential equations using ADM has been developed[5, 6, 7, 8, 9, 10].

In the recent times, technque of approximating the solutions of differential equations by orthogonal polynomial has become extreemly popular. In that line of thoughts, Chebyshev polynomial is one of the important polynomials which is widely used[11, 12, 13] to solve various linear and nonlinear differential equations having physical significance.

In this paper, we have presented a simplified technique to handle highly complicated source term present in nonlinear fractional differential equations. Adomian Decomposition method is applied to these equations by using Taylor series expansion and Chebyshev polynomial representation of source terms.

2. PRELIMINARIES

2.1 BASIC DEFINITION OF FRACTIONAL CALCULUS

In the section we have quoted some of the important definitions of fractional calculus [14].

Definition 2.1 The Riemann-Liouville fractional integral operator J^{α} of order $\alpha \geq 0$ is defined as

$$J^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}y(t)dt,\ (\alpha > 0, t > 0)$$

such that $J^0 y(x) = y(x)$

Definition 2.2 The fractional derivative of order α in Caputo sense with $n-1 < \alpha < n$ $of\ f(t), t > 0$ is defined by

$$D^{\alpha}f(t) = J^{n-\alpha}D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{(\alpha-n+1)}}dt$$

2.2 CHEBYSHEV POLYNOMIAL

$T_i(x)$ is the first kind of orthogonal Chebyshev polynomial,

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

and in general $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),\ k >= 1.$

Applying the method described in [14] we have

$$y_0 = \sum_{i=0}^{\alpha} y^{(k)}(0)\frac{x^k}{\Gamma(k+1)} + J^{\alpha}(a_0 T_0(x))$$

2.3 ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method has been used in [9-14] to solve effectively, easily and accurately a large class of linear and non-linear ordinary, partial, deterministic or stochastic fractional differential equations with approximate solutions which converge rapidly to accurate solutions. In this part, the concept of Adomian decomposition method is presented. For this reason, we consider the differential equation

$$Ly(x) + Ry(x) + Ny(x) = g(x) \tag{1}$$

With initial value $y^i(0) = \alpha_i,\ i = 0,1,2,\ldots n$

Here L is and invertible linear operator, R is the linear operator and N is a nonlinear operator Using the inverse operator L^{-1} to both sides of Eq.1, we obtain

$$y = \phi(x) + L^{-1}[g(x) - Ry(x) - Ny(x)]$$

$$y = \phi(x) + L^{-1}g(x) - L^{-1}Ry(x) - L^{-1}Ny(x)$$

Where $\phi(x)$ arises from the given initial condition.The ADM suggests that the solution y(x) may be decomposed by the infinite series of components

$$y(x) = \sum_{i=0}^{n} y_n(x) \tag{2}$$

and the non-linear operator $N(y)$ can be decomposed by an infinite series of polynomials given by

$$N(y) = \sum_{i=0}^{n} A_n(x) \tag{3}$$

and the A_n are the so-called Adomian Polynomials of $y_0, y_1, \ldots \mathrm{y_n}$ defined by

$$A_n(y_0, y_1, \ldots \mathrm{y_n}) = [\frac{1}{\mathrm{n!}}\frac{\mathrm{d^n}}{\mathrm{d\lambda^n}}\mathrm{F}(\sum_{\mathrm{i=0}}^{\mathrm{n}} \lambda^{\mathrm{i}}\mathrm{y_i}(\mathrm{x})]_{\lambda=0} \tag{4}$$

considering the fractional differential equation of the form

$$D^{\alpha}y(x) + Ry(x) + Ny(x) = g(x) \tag{5}$$

with initial condition $y^i(o) = \alpha_i,\ i = 0,1,2,\ldots n$ and $0 < n < \alpha$, where D^{α} represents fractional derivative operator,R is the linear operator, N is the nonlinear operator and g(x) is the source term.

The method is based on applying the Riemann-Liouville fractional integral operator J^{α}. Using the inverse on equation 5 , we get

$$y(x) = \sum_{k=0}^{\alpha} y^{(k)}(0)\frac{x^k}{\Gamma(k+1)} + J^{\alpha}[g(x)] - J^{\alpha}[Ry(x)] - J^{\alpha}[Ny(x)] \tag{6}$$

We write $N(y) = \sum_{i=0}^{n} A_n(x)$ and $y(x) = \sum_{i=0}^{n} y_n(x)$, where the components of $A_n(x)$, called adomain polynomials for each i, $A_i(x)$ depends on $y_0, y_1, \ldots \mathrm{y_n}$ only.

Substituting 2 and 3 in Eq.6, we obtain

$$y_0 = \sum_{i=0}^{\alpha} y^{(k)}(0)\frac{x^k}{\Gamma(k+1)} + J^\alpha g(x)$$

$$y_1 = J^\alpha[Ry_0] - J^\alpha[A_0]$$

$$y_2 = J^\alpha[Ry_1] - J^\alpha[A_1]$$

$$\vdots$$

$$y_{k+1} = J^\alpha[Ry_k] - J^\alpha[A_k]$$

The required expressions for $A_i(x)$'s are

$$A_0 = F(y_0)$$

$$A_1 = y_1 F'(y_0)$$

$$A_2 = y_2 F'(y_0) + \frac{1}{2} y_1^2 F''(y_0)$$

$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) \frac{1}{2} y_1^3 F'''(y_0)$$

$$\vdots$$

If the series converges, we can see that

$$y = \lim_{n->\infty} \sum_{i=0}^{n} y_i(x)$$

And where F(y)=N(y).

To perform ADM for complicated source term g(x), we shall express it in Chebyshev polynomials as

$$g(x) = \sum_{i=0}^{m} a_i T_i(x)$$

where $T_i(x)$ is the first kind of orthogonal Chebyshev polynomial [11].

In the following section, we have applied the above method to two nolinear fractional differential equations.

3. APPLICATIONS

Example 3.1: We consider the fractional differential equation

$$y^\alpha + xy' + x^2 y^3 = (2+6)x^2)e^{x^2} + x^2 e^{3x^2} \qquad (7)$$

$$y(0) = 1 \quad y'(0) = 0. \; 1 \le \alpha \le 2$$

Exact solution at $\alpha = 2$ is $u(x) = e^{x^2}$

Solution: According to Eq.5 we have

$R = x\frac{d}{dx}$, $Ny = x^2 y^3$ and $g(x) = (2+6)x^2)e^{x^2} + x^2 e^{3x^2}$.

Since $Ny = x^2 y^3$, the Adomain polynomials are,

$$A_0 = x^2 y_0^3$$

$$A_1 = x^2(3y_0^2y_1)$$

$$A_2 = x^2(3y_0^2y_2 + 3y_0y_1^2)$$

$$A_3 = x^2(3y_0^2y_3 + 6y_0y_1y_2 + y_1^3)$$

case(i): The Taylor series expansion of g(x) is

$$g(x) = 2 + 9x^2 + 10x^4 + \frac{47x^6}{6}$$

Applying the ADM method described in Section 2.3, we get

$$y_0 = 2x^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{9x^2}{\Gamma(\alpha+3)} + 60x^4\left(\frac{2}{\Gamma(\alpha+5)} + \frac{47x^2}{\Gamma(\alpha+7)}\right)\right) + 1$$

$$A_0 = t^2\left(2t^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + \frac{9t^2}{\Gamma(\alpha+3)} + 60t^4\left(\frac{2}{\Gamma(\alpha+5)} + \frac{47t^2}{\Gamma(\alpha+7)}\right)\right) + 1\right)^3$$

$$y_1 = -\frac{2x^{2\alpha}(3(\Gamma(\alpha+1)(\frac{940x^4}{\Gamma(2\alpha+7)}+\frac{40x^2}{\Gamma(2\alpha+5)}+\frac{3}{\Gamma(2\alpha+3)})+\Gamma(\alpha)(\frac{5640x^4}{\Gamma(2\alpha+7)}+\frac{160x^2}{\Gamma(2\alpha+5)}+\frac{6}{\Gamma(2\alpha+3)})x^2+\frac{4^{-\alpha}\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}}{\Gamma(\alpha)} -$$

$$\frac{1}{\Gamma(\alpha+1)^3\Gamma(\alpha+3)^3\Gamma(\alpha+5)^3\Gamma(\alpha+7)^3}(2x^{\alpha+2}(3\Gamma(\alpha+1)^2\Gamma(\alpha+3)\Gamma(\alpha+5)\Gamma(\alpha+7)$$

$$(36\Gamma(\alpha+3)\Gamma(\alpha+5)\Gamma(\alpha+7)\frac{240\Gamma(\alpha+7)\Gamma(3\alpha+9)x^{\alpha+4}}{\Gamma(4\alpha+9)} + \Gamma(\alpha+5)(\frac{\Gamma(\alpha+7)\Gamma(2\alpha+5)}{\Gamma(3\alpha+5)} +$$

$$\frac{5640\Gamma(3\alpha+11)x^{\alpha+6}}{\Gamma(4\alpha+11)})\ldots)))$$

And so on.

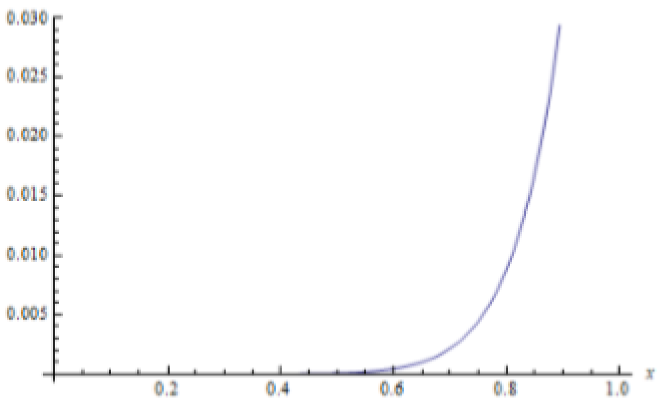

Figure 1 : Absolute norm $\| exact - y_{Taylor} \|$ in case(i) at $\alpha = 2$ for Example 1

case(ii): Now let M=6, since L^{-1} Riemann-Liouville fractional integral operator and the Chebyshev polynomials of g(x) is

$$g(x) = 2.03164 - 2.89636x + 51.4781x^2 - 226.976x^3 + 560.267x^4 - 623.301x^5 + 281.173x^6$$

Applying the methods described in section 2.3 yields

$$y_0 = x^\alpha\left(\frac{2.03164}{\Gamma(\alpha+1)}\right) + x\left(x\left(\frac{102.956}{\Gamma(\alpha+3)} + \frac{202445.x^4}{\Gamma(\alpha+7)} - \frac{74796.1x^3}{\Gamma(\alpha+6)} + \frac{13446.4x^2}{\Gamma(\alpha+5)} - \frac{1361.86x}{\Gamma(\alpha+4)}\right) - \frac{2.89636}{\Gamma(\alpha+2)}\right) +1$$

$$A_0 = t^2\left(t^\alpha\left(\frac{2.03164}{\Gamma(\alpha+1)} + t\left(t\left(\frac{102.956}{\Gamma(\alpha+3)} + \frac{202445.t^4}{\Gamma(\alpha+7)} - \frac{74796.1t^3}{\Gamma(\alpha+6)} + \frac{13446.4t^2}{\Gamma(\alpha+5)} - \frac{1361.86t}{\Gamma(\alpha+4)}\right) - \frac{2.89636}{\Gamma(\alpha+2)}\right)\right) + 1\right)^3 \vdots$$

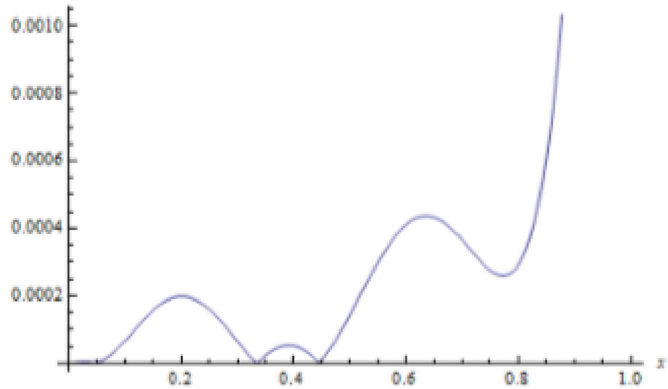

Figure 2: Absolute norm $\| exact - y_{Chebyshev} \|$ in case(ii) at $\alpha = 2$ for Example 1

Example 3.2: Let us consider the equation

$$y^\alpha + yy' = x\sin(x^2) - 4x^2\sin(x^2) + 2\cos(x^2) \quad (8)$$

$y(0) = 0,\ \ y'(0) = 0.\ 1 \le \alpha \le 2$

Exact solution at $\alpha = 2$ is $y(x) = \sin(x^2)$

Solution According to Eq.5 we have

$$D^\alpha y(x) + Ry(x) + Ny(x) = g(x)$$

where $R = 0$, $Ny = yy'$ and $g(x) = x\sin(x^2) - 4x^2\sin(x^2) + 2\cos(x^2)$.

in addition, $F(y) = Ny = yy'$, the Adomain polynomials are,

$A_0 = y_0y_0'$

$A_1 = y_1y_0^2 + y_0y_1^2$

$A_2 = y_2y_0^2 + y_1y_1^2 + y_0y_2^2$

$A_3 = y_3y_0^2 + y_2y_1^2 + y_1y_2^2 + y_0y_3^2$

case(i): Now let M=8, since L^{-1} Riemann-Liouville fractional integral operator and the Taylor series of g(x) is

$$g(x) = 2 + 2x^3 - 5x^4 - \frac{4x^7}{3} + \frac{3x^8}{4}$$

So, by Eq.2.3, we have

$$y_0 = 2x^\alpha\left(\frac{1}{\Gamma(\alpha+1)} + 6x^3\left(\frac{1}{\Gamma(\alpha+4)} + 280x^4\left(\frac{9x}{\Gamma(\alpha+9)} - \frac{2}{\Gamma(\alpha+8)}\right) - \frac{10x}{\Gamma(\alpha+5)}\right)\right)$$

$$A_0 = \frac{1}{\Gamma(\alpha+1)^2\Gamma(\alpha+4)^2\Gamma(\alpha+5)^2\Gamma(\alpha+8)^2\Gamma(\alpha+9)^2}4t^{2\alpha-1}(\Gamma(\alpha+4)\Gamma(\alpha+5)\Gamma(\alpha+8)\Gamma(\alpha+9) + 6t^3\Gamma(\alpha+1)(\Gamma(\alpha+5)\Gamma(\alpha+8)\Gamma(\alpha+9) + 10\Gamma(\alpha+4)(28t^4\Gamma(\alpha+5)(9t\Gamma(\alpha+8) - 2\Gamma(\alpha+9)) - t\Gamma(\alpha+8)\Gamma(\alpha+9))))(\alpha\Gamma(\alpha+4)\Gamma(\alpha+5)\Gamma(\alpha+8)\Gamma(\alpha+9) + 6t^3\Gamma(\alpha+1)((\alpha+3)\Gamma(\alpha+5)\Gamma(\alpha+8)\Gamma(\alpha+9) + 10t\Gamma(\alpha+4)(28t^3\Gamma(\alpha+5)(9(\alpha+8)t\Gamma(\alpha+8) - 2(\alpha+7)\Gamma(\alpha+9)) - (\alpha+4)\Gamma(\alpha+8)\Gamma(\alpha+9))))$$

$\vdots$

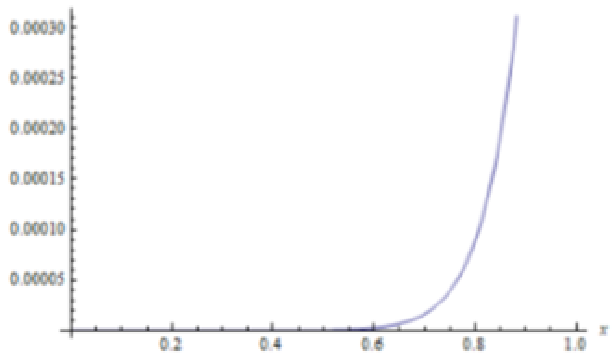

Figure 3: Absolute norm $\|\ exact - y_{Taylor}\ \|$ in case(i) at $\alpha = 2$ for Example 2

case(ii): Now let M=6, since L^{-1} Riemann-Liouville fractional integral operator and the Chebyshev polynomials of g(x) is

$$g(x) = 2.00062 - 0.0580807x + 0.874789x^2 - 2.8117546655765 11x^3 + 7.00329x^4 - 13.7887x^5 + 5.40309x^6$$

The following figure shows the error curve.

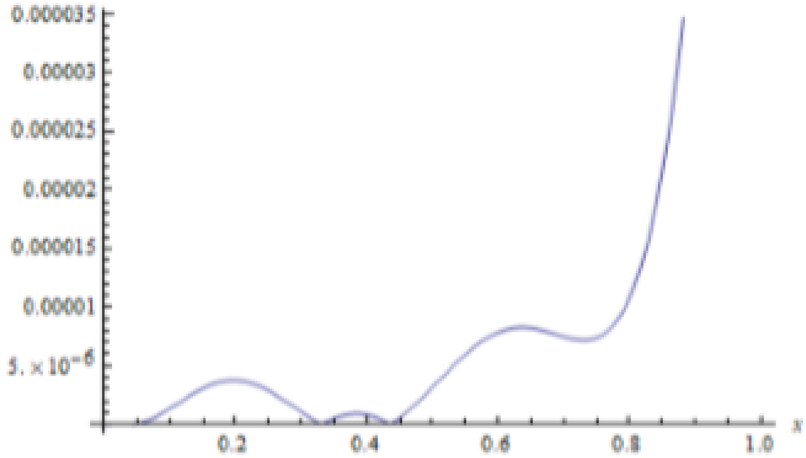

Figure 4: Absolute norm $\| exact - y_{Chebyshev} \|$ in Case(ii) at $\alpha = 2$ for Example 2

4. CONCLUSION:

In this work, we have applied Adomian Decomposition method with Taylor series and Chebyshev polynomials. On applying the method to the examples with inhomoheneous source terms we can conclude from Fig.1, Fig.2, Fig.3 and Fig.4 that Chebyshev polynomial series representation of inhomogeneous source term produces better results than Taylor series expansion.

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