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Efficient Product-type Exponential Estimator for Population Variance

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ABSTRACT

We in this paper propose a new exponential product type estimator for estimating the population variance using auxiliary information. To the first order of approximation, i.e., to $o(n^{-1})$, the expressions for the bias and the mean square error of the proposed exponential product-type estimator have been derived. The optimum value of the characterizing scalar, which minimizes the MSE of proposed estimator, has been obtained. With this optimum value, the expression for minimum MSE of the proposed estimator has been arrived at. The proposed estimator has been compared theoretically with simple variance, traditional product estimator and exponential product-type estimator and it is found that, under practical conditions, the proposed estimator fares better than its competing estimators. A numerical study is done to demonstrate the practical use of different estimation formulae and empirically demonstrate the performance of the constructed estimators.

KEYWORDS: Auxiliary variable, single-phase sampling, bias, mean square error.

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INTRODUCTION

In survey sampling, the utilization of auxiliary information is frequently acknowledged to enhance the accuracy of the estimation of population characteristics. Estimation of the finite population variance has great significance in various fields such as in matters of health, variation in body temperature, pulse beat and blood pressure etc. Using auxiliary information, we, in this paper, introduce a new estimator, which fares better than its competing estimators.

Consider a finite population of size N, arbitrarily labelled 1, 2....N. Let y_i and x_i be, respectively, the values of the study variable y and the auxiliary variable x, in respect of the ith unit (i=1, 2,… N) of the population. When the auxiliary variable x is negatively correlated with the study variable y and S_x^2 , the population variance of x is known, product method of estimation is usually invoked to estimate the population variance S_y^2 of the study variable. The product method of estimation investigated by Robson[1] and Murthy[2], is quite effective.

NOTATIONS AND SOME EXISTING ESTIMATORS

In simple random sampling without replacement, we know that the sample variance s_y^2 provides an unbiased estimator of the population variance S_y^2,

where $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})^2$

$and\ s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$.

Accordingly, we define

$$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{X})^2$$

$and\ s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$,

as the population and sample variances, respectively, for the auxiliary variable x.

Let $e_0 = \frac{s_y^2 - S_y^2}{S_y^2}, i.e., s_y^2 = S_y^2(1 + e_0)$,

$$e_1 = \frac{s_x^2 - S_x^2}{S_x^2}, i.e., s_x^2 = S_x^2(1 + e_1),$$

such that E(e_0)=E(e_1)=0, E(e_0^2) $=\frac{1}{n}(\lambda_{40} - 1)$, E($e_1^2$) = $\frac{1}{n}(\lambda_{04} - 1)$ and E($e_0 e_1$) = $\frac{1}{n}(\lambda_{22} - 1)$,

where $\lambda_{pq} = \frac{\mu_{pq}}{\mu_{20}^{p/2}\mu_{02}^{q/2}}$

and $\mu_{pq} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \overline{Y})^{p}(x_i - \overline{X})^{q}$; (p, q) being non-negative integers, μ_{02},μ_{20} are the second order moments of x and y, respectively, and $C_x = \frac{S_x}{\overline{X}}$ is the coefficient of variation for auxiliary variable X. With the above notations, the variance of the estimator s_y^2 is expressed as

$$\mathrm{V}(s_y^2) = \frac{1}{n} S_y^4(\lambda_{40} - 1). \tag{1}$$

Customary product type estimator for estimating the population variance of the study variable is

$$s_{y_P}^2 = \frac{s_y^2 s_x^2}{S_x^2}, \tag{2}$$

whose bias and mean square error, up to the first order of approximation ,i.e., to o (n^{-1}) are respectively,

$$\mathrm{B}(s_{y_P}^2) = \frac{1}{n} S_y^2 \, (\lambda_{22} - 1) \tag{3}$$

and

$$\mathrm{MSE}(s_{y_P}^2) = \frac{S_y^4}{n}[\lambda_{40} + \lambda_{04} + 2\lambda_{22} - 4]. \tag{4}$$

Singh et. al.[3] suggested product-type exponential estimator for population variance in single-phase sampling as

$$s_{y_{Pe}}^2 = s_y^2 \exp\left[\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right], \tag{5}$$

whose bias and mean square error up to first order of approximation, i.e., to o (n^{-1}) are, respectively,

$$\mathrm{B}(s_{y_{Pe}}^2) = \frac{S_y^2}{n}\left[-\frac{\lambda_{04}}{8} + \frac{\lambda_{22}}{2} - \frac{3}{8}\right] \tag{6}$$

and

$$\mathrm{MSE}(s_{y_{Pe}}^2) = \frac{S_y^4}{n}\left[\lambda_{40} + \frac{\lambda_{04}}{4} + \lambda_{22} - \frac{9}{4}\right]. \tag{7}$$

PROPOSED PRODUCT-TYPE EXPONENTIAL ESTIMATOR

We propose a new product-type exponential estimator for estimating the population variance S_y^2, as

$$s'^2_{y_{Pe}} = s_y^2 \exp[\frac{\beta(s_x^2 - S_x^2)}{s_x^2 + S_x^2}], \tag{8}$$

where, β is a suitably chosen pre-assigned constant. It may be noted here that if $\beta = 1$, the new estimator reduces to usual product-type estimator due to Singh et.al.[2011]

Substituting the value of s_y^2, s_x^2 in the expression (8), we get

$$s'^2_{y_{Pe}} = s_y^2 \exp[\frac{\beta(S_x^2 - S_x^2 + S_x^2 e_1)}{S_x^2 + S_x^2 + S_x^2 e_1}]$$

$$= s_y^2 \exp[\frac{\beta(e_1 S_x^2)}{S_x^2(2+e_1)}]$$

$$= S_y^2(1 + e_0)\exp[\beta(e_1)(2 + e_1)^{-1}].$$

$$= S_y^2(1 + e_0)\exp[\beta\frac{e_1}{2}(1 + \frac{e_1}{2})^{-1}].$$

$$= S_y^2(1 + e_0)\exp[1 + \beta(\frac{e_1}{2})(1 - \frac{e_1}{2} + \frac{e_1^2}{4} - \frac{e_1^3}{8}) + \beta^2\frac{e_1^2}{8}(1 - \frac{e_1}{2} + \frac{e_1^2}{4} - \frac{e_1^3}{8})^2]$$

Retaining terms only up to 2nd degree, we arrive at

$$s'^2_{y_{Pe}} = S_y^2(1 + e_0)\exp[1 + \beta(\frac{e_1}{2}) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8}]$$

$$= S_y^2[1 + \beta\left(\frac{e_1}{2}\right) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8} + e_0 + \beta\left(\frac{e_0 e_1}{2}\right)]$$

$$= S_y^2[1 + \beta\left(\frac{e_1}{2}\right) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8} + e_0 + \beta\left(\frac{e_0 e_1}{2}\right)] \tag{9}$$

The bias of the proposed exponential product estimator, to the first degree of approximation, i.e., to o (n^{-1}), is

$$B(s'^2_{y_{Pe}}) = E(s'^2_{y_{Pe}}) - S_y^2$$

$$= S_y^2 E[1 + \beta\left(\frac{e_1}{2}\right) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8} + e_0 + \beta\left(\frac{e_0 e_1}{2}\right)] - S_y^2$$

$$= S_y^2[E(1) + \frac{\beta}{2}E(e_1) - \frac{\beta}{4}E(e_1^2) + \frac{\beta^2}{8}E(e_1^2) + E(e_0) + \frac{\beta}{2}E(e_0 e_1)]$$

$$= \frac{S_y^2}{n}[-\frac{\beta}{4}(\lambda_{04} - 1) + \frac{\beta^2}{8}(\lambda_{04} - 1) + \frac{\beta}{2}(\lambda_{22} - 1)]. \tag{10}$$

The mean square error of the proposed exponential product estimator, to the first degree of approximation, i.e., to o (n^{-1}), has been derived as follows:

$MSE(s'^2_{yPe}) = E[s'^2_{yPe} - s^2_y]^2$

$$= E[S_y^2\left(1 + \beta\left(\frac{e_1}{2}\right) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8} + e_0 + \frac{\beta}{2}(e_0e_1)\right) - S_y^2]^2$$

$$= s_y^4 E[\left(\beta\left(\frac{e_1}{2}\right) - \beta\frac{e_1^2}{4} + \beta^2\frac{e_1^2}{8} + e_0 + \frac{\beta}{2}(e_0e_1)\right)]^2$$

$$= S_y^4 E[\left(\frac{\beta^2}{4}e_1^2 + e_0^2 + \beta e_0 e_1\right)]$$

$$= S_y^4[\left(\frac{\beta^2}{4}E(e_1^2) + E(e_0^2) + \beta E(e_0e_1)\right)]$$

$$= S_y^4[\frac{\beta^2}{4n}(\lambda_{04} - 1) + \frac{1}{n}(\lambda_{40} - 1) + \frac{\beta}{n}(\lambda_{22} - 1)]$$

$$= \frac{S_y^4}{n}\left[\frac{\beta^2}{4}(\lambda_{04} - 1) + (\lambda_{40} - 1) + \beta(\lambda_{22} - 1)\right]. \tag{11}$$

With a view to determining the most suitable value of β, to be called β_{opt}, we minimize the mean square error subject to variation in β, implying thereby that

$$\frac{\partial MSE\ (s'^2_{yPe})}{\partial\beta} = 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{\beta^2}{4}(\lambda_{04} - 1) + (\lambda_{40} - 1) + \beta(\lambda_{22} - 1)] = 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{2\beta}{4}(\lambda_{04} - 1) + (\lambda_{22} - 1)] = 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{\beta}{2}(\lambda_{04} - 1) + (\lambda_{22} - 1)] = 0$$

$$\Rightarrow \frac{\beta}{2}(\lambda_{04} - 1) + \lambda_{22} - 1 = 0$$

$$\Rightarrow \frac{\beta}{2}(\lambda_{04} - 1) = 1 - \lambda_{22}$$

$$\Rightarrow \beta(\lambda_{04} - 1) = 2(1 - \lambda_{22})$$

$$\Rightarrow \beta = \frac{2(1-\lambda_{22})}{(\lambda_{04}-1)}.$$

Thus, $\beta_{opt} = \frac{2(\lambda_{22}-1)}{(1-\lambda_{04})}$. (12)

Substituting this value of β in the expression for MSE ($s'^2_{y_{Pe}}$), i.e., in (11), we arrive at the minimum value of MSE ($s'^2_{y_{Pe}}$), which is expressed as

$$MSE_{opt}(s'^2_{y_{Re}}) = \frac{S_y^4}{n}[\frac{2\lambda_{22}-\lambda_{22}^2-1)}{\lambda_{04}-1} + \lambda_{40} - 1]. \quad (13)$$

On comparison of (13) with (1), the following results can be arrived at

$$MSE_{opt}(s'^2_{y_{Pe}}) - \mathrm{V}\,(s_y^2) < 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{2\lambda_{22}-\lambda_{22}^2-1)}{\lambda_{04}-1} + \lambda_{40} - 1] - \frac{1}{n}[S_y^4(\lambda_{40} - 1)] < 0$$

$$\Rightarrow \lambda_{04} > 1. \quad (14)$$

On comparison of (13) with (4), the following results can be arrived at

$$MSE_{opt}(s'^2_{y_{Pe}}) - \mathrm{MSE}\,(s^2_{y_P}) < 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{2\lambda_{22}-\lambda_{22}^2-1)}{\lambda_{04}-1} + \lambda_{40} - 1] - \frac{S_y^4}{n}[\lambda_{40} + \lambda_{04} + 2\lambda_{22} - 4] < 0$$

$$\Rightarrow \frac{(1-\lambda_{22})^2}{(\lambda_{04}-1)} + \lambda_{04} + 2\lambda_{22} > 3. \quad (15)$$

Comparing (13) with (7), we get

$$MSE_{opt}(s'^2_{y_{Pe}}) - MSE(s^2_{y_{Pe}}) < 0$$

$$\Rightarrow \frac{S_y^4}{n}[\frac{2\lambda_{22}-\lambda_{22}^2-1)}{\lambda_{04}-1} + \lambda_{40} - 1] - \frac{S_Y^4}{n}[\lambda_{40} + \frac{\lambda_{04}}{4} + \lambda_{22} - \frac{9}{4}] < 0$$

$$\Rightarrow \frac{1+\lambda_{22}^2-2\lambda_{22}}{\lambda_{04}-1} + \frac{\lambda_{04}}{4} + \lambda_{22} > \frac{5}{4}. \quad (16)$$

(1) The newly proposed estimator $s'^2_{y_{Pe}}$ performs better than the simple variance estimator of population variance s_y^2 if

$$\lambda_{04} > 1.$$

(2) The newly proposed estimator $s'^2_{y_{Pe}}$ performs better than the traditional product-type estimator for variance s^2_{yp} if

$$\frac{(1-\lambda_{22})^2}{(\lambda_{04}-1)} + \lambda_{04} + 2\lambda_{22} > 3.$$

(3) The newly proposed estimator $s'^2_{y_{Pe}}$ performs better than the product-type exponential estimator due to Singh et. al.[3] for variance, i.e., $s^2_{y_{Pe}}$ if

$$\frac{1+\lambda_{22}^2-2\lambda_{22}}{\lambda_{04}-1} + \frac{\lambda_{04}}{4} + \lambda_{22} > \frac{5}{4}.$$

EMPIRICAL STUDY

With a view to study the theoretical findings, we have chosen the following examples based on real data:

Example 1: We refer to Weisberg[4] PP- 31-35, wherein the following information is found:

Y: Fuel consumption, gallons per person.

X: TAX, cents per gallon

Table 1: Parameters and their values

Sl.No.	Parameter	Value of the Parameter
1	N	48
3	$\bar{X}$	0.570354
4	$\bar{Y}$	576.7708
5	ρ_{yx}	-0.4512
6	λ_{40}	5.3604
7	λ_{04}	3.3096
8	λ_{22}	2.9198

$$\lambda_{04} > 1$$

$$\Rightarrow 3.3096 > 1.$$

$$\frac{(1-\lambda_{22})^2}{(\lambda_{04}-1)} + \lambda_{04} + 2\lambda_{22} > 3$$

$$\Rightarrow 5.89095 > 3.$$

$$\frac{1+\lambda_{22}^2-2\lambda_{22}}{\lambda_{04}-1} + \frac{\lambda_{04}}{4} + \lambda_{22} > \frac{5}{4}$$

$$\Rightarrow 5.34298 > \frac{5}{4}.$$

Thus, we find that the condition (14), (15) and (16) are satisfied.

The MSEs of the competing estimators have been computed and presented in Table 2

Table 2: MSE of the competing estimator

Sl. No.	CompetingEstimator	MSE/ $\frac{S_y^4}{n}$
1	s_y^2	4.3604
2	s_{yP}^2	10.5096
3	s_{yPe}^2	6.8576
4	$s_{yPe}'^2$	2.7646

The percentage gain in efficiency of the proposed estimator, $s_{yPe}'^2$ over the competing estimators s_y^2, s_{yP}^2 and s_{yPe}^2 has been given in the table 3

Table 3

Sr. No.	CompetingEstimator	Percentage Gain in Efficiency(PGE)
1	s_y^2	57.7226
2	s_{yP}^2	280.1490
3	s_{yPe}^2	148.0503

It is clear from the above table that the newly proposed estimator $s_{yPe}'^2$ performs better than the competing estimators.

Example 2: We refer to Swain[5], PP-285-287, wherein the following information is found:

Y: Mean yield of rice per plant

X: percentage of sterility

Table 4: Parameters and their values

Sl. No.	Parameter	Value of the Parameter
1	N	50
3	$\overline{X}$	18.762
4	$\overline{Y}$	12.839
5	ρ_{yx}	-0.25103
6	λ_{40}	3.828497
7	λ_{04}	2.474604
8	λ_{22}	1.60349

$$\lambda_{04} > 1$$

$$\Rightarrow 2.474604 > 1.$$

$$\frac{(1-\lambda_{22})^2}{(\lambda_{04}-1)} + \lambda_{04} + 2\lambda_{22} > 3$$

$$\Rightarrow 5.92856 > 3.$$

$$\frac{1+\lambda_{22}^2-2\lambda_{22}}{\lambda_{04}-1} + \frac{\lambda_{04}}{4} + \lambda_{22} > \frac{5}{4}$$

$$\Rightarrow 2.46912 > \frac{5}{4}.$$

Thus, we find that the condition (14), (15) and (16) are satisfied.

The MSEs of the competing estimators have been computed and presented in Table 5

Table 5: MSE of the competing estimator

Sl. No.	$CompetingEstimator$	$MSE/\frac{S_y^4}{n}$
1	s_y^2	2.828497
2	s_{yP}^2	5.51008
3	s_{yPe}^2	3.800637
4	$s_{yPe}'^2$	2.5815

The percentage gain in efficiency of the proposed estimator, $s_{yPe}'^2$ over the competing estimators s_y^2, s_{yP}^2 and s_{yPe}^2 has been given in the table 6

Table 6:

Sl. No.	$CompetingEstimator$	Percentage Gain in Efficiency(PGE)
1	s_y^2	**9.5676**
2	s_{yP}^2	**113.4448**
3	s_{yPe}^2	**47.2259**

It is clear from the above table that the newly proposed estimator $s_{yPe}'^2$ performs better than the competing estimators.

V. CONCLUSION

From Tables 3 & 6, it is clear that the proposed product-type exponential estimator for estimating the population variance perform better than its competing estimators under conditions that hold good in practice.

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